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CONTENTS

Geometric Figures in Affine Space. 1
By Domina Eberle Spencer

Measurable Functions. 24
By Philip Franklin

Table of $f_n(x) = \frac{n!}{(x/2)^n} J_n(x)$. 45
By the Mathematical Tables Project

A Boundary Value Problem in Plane Plasticity. 61
By N. Coburn

An Improvement on the G.C.D. Method for Complex Roots. 69
By Frank L. Hitchcock

Table of Coefficients for Inverse Interpolation with Advancing Differences. 75
By Herbert E. Salzer

The Tensor Representation of the Figures of Study's "Geometrie der Dynamen". 103
By Domina Eberle Spencer

A Boundary Value Problem in Plane Plasticity for the Coulomb Yield Condition. 117
By N. Coburn

Transversality in Higher Space. 126
By Jacques Dutka

On the Foundations of Electrical Network Theory. . . 134
By W. H. Ingram and O. M. Cramlet

Coefficients for Interpolation Within a Square Grid in the Complex Plane. 156
By Arnold N. Lowan and Herbert E. Salzer

Surfaces of Maximum Shearing Stress. 167
By T. Y. Thomas

MATHEMATICS GRADUATE LIBRARY

On the Asymptotic Solution of Boundary Value Problems for Ordinary Differential Equations Containing a Parameter. 173
By WOLFGANG WASOW

On the Theory of Bending of Elastic Plates. 184
By ERIC REISSNER

Note on the Theorem of the Symmetry of the Stress Tensor. 192
By ERIC REISSNER

The Wiener Measure of Hilbert Neighborhoods in the Space of Real Continuous Functions 195
By R. H. CAMERON AND W. T. MARTIN

Table of Coefficients for Differences in Terms of the Derivatives. 210
By HERBERT E. SALZER

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GEOMETRIC FIGURES IN AFFINE SPACE*

By Domina Eberle Spencer

In the past forty years Study's "Geometrie der Dynamen" has not received the attention it deserves. It is a study of systems of free and attached vectors which occur so frequently in physical problems. If the tensor coordinatization of these geometric figures can be found it will make a contribution to geometry and applied mathematics comparable with that made by vector analysis.

In order to find the tensor coordinatization of the figures of Study one must determine their place in the structure of geometry. A cursory survey of the "Geometrie der Dynamen" shows that, although Study worked in metric space, his figures are inherently affine. Thus an investigation of the geometric figures of affine space is desirable and should lead to a more basic understanding of Study's work. Since his figures are all obtained as combinations of points and planes a similar procedure is indicated here. Insofar as possible the tensor notation introduced by Schouten and Struik in their "Einführung in die neueren Methoden der Differentialgeometrie" will be used.**

In the present paper, the geometric figures represented by tensors of valence one are discussed. Figures represented by tensors of valence two are formed by simple combinations of these. The paper develops several geometric figures that have not been discussed previously. Among the combinations are found the affine ancestors of all the figures of Study.

Elements of Projective Space

Figures in projective space are those geometric figures that have invariant properties under projective transformations. Of particular interest will be the point, the line, and the plane. It is convenient to let Greek indices run through the range 0, 1, 2, 3, and Latin indices take on the values 1, 2, 3.

A point is represented by a contravariant pseudotensor $\bar{x}^\sigma$ whose law of transformation is

$$\bar{x}^{\sigma'} = \tau A^{\sigma'}_{\sigma} \bar{x}^{\sigma}. \tag{1}$$

$$\Delta = |A^{\sigma'}_{\sigma}| \neq 0, \ \tau \text{ any scalar} \neq 0.$$

* Presented at the Summer Meetings of the American Mathematical Society at Poughkeepsie, N. Y. in September, 1942. The paper is an abstract of part of a doctorate thesis which was accepted by the Massachusetts Institute of Technology in May 1942, "A Tensor Interpretation of Study's *Geometrie der Dynamen*." The author gratefully acknowledges the valuable suggestions of Prof. D. J. Struik.

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E. Study, Geometrie der Dynamen, Leipzig, 1903, Teubner, 603 pp.

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As the principle of duality of projective geometry would indicate, a plane is represented by a covariant pseudotensor $\tilde{u}_\lambda$ whose law of transformation is

$$\tilde{u}_{\lambda'} = \tau A^{\lambda}_{\lambda'} \tilde{u}_\lambda. \tag{2}$$

A line may be defined in terms of either two points or two planes. If it is defined by two distinct points $\tilde{x}^\kappa$ and $\tilde{y}^\lambda$, its tensor representation is $\tilde{p}^{\kappa\lambda}$, where

$$\tilde{p}^{\kappa\lambda} = \tilde{x}^{[\kappa}\tilde{y}^{\lambda]} = \tilde{x}^\kappa \tilde{y}^\lambda - \tilde{x}^\lambda \tilde{y}^\kappa \tag{3}$$

and the law of transformation is

$$\tilde{p}^{\kappa'\lambda'} = \tau^2 A^{\kappa'}_{\kappa} A^{\lambda'}_{\lambda} \tilde{p}^{\kappa\lambda}. \tag{4}$$

This tensor also satisfies the Plücker identity:

$$\tilde{p}^{[\kappa\lambda}\tilde{p}^{\mu\nu]} \equiv 0. \tag{5}$$

If the line is defined in terms of two planes $\tilde{u}_\kappa$ and $\tilde{v}_\lambda$, its tensor representation is $\tilde{p}_{\kappa\lambda}$, where

$$\tilde{p}_{\kappa\lambda} = \tilde{u}_{[\kappa}\tilde{v}_{\lambda]} \tag{6}$$

and the law of transformation is

$$\tilde{p}_{\kappa'\lambda'} = \tau^2 A^{\kappa}_{\kappa'} A^{\lambda}_{\lambda'} \tilde{p}_{\kappa\lambda}. \tag{7}$$

The Plücker identity is now written

$$\tilde{p}_{[\kappa\lambda}\tilde{p}_{\mu\nu]} \equiv 0. \tag{8}$$

In order to properly associate these two representations of a line, the E pseudotensor is introduced. It can be defined as follows:

$$\begin{aligned} \tilde{E}^{\kappa\lambda\mu\nu} &= 0 && \text{if any pair of indices are equal,} \\ &= 1 && \text{if } \kappa\lambda\mu\nu \text{ is an even permutation of } 0, 1, 2, 3 \\ &= -1 && \text{if } \kappa\lambda\mu\nu \text{ is an odd permutation of } 0, 1, 2, 3. \end{aligned} \tag{9}$$

$$\begin{aligned} \tilde{E}_{\kappa\lambda\mu\nu} &= 0 && \text{if any pair of indices are equal} \\ &= 1 && \text{if } \kappa\lambda\mu\nu \text{ is an even permutation of } 0, 1, 2, 3 \\ &= -1 && \text{if } \kappa\lambda\mu\nu \text{ is an odd permutation of } 0, 1, 2, 3. \end{aligned} \tag{10}$$

This permits a one-to-one correspondence between contravariant and covariant co-ordinates of a line by the equations

$$\begin{aligned} \tilde{p}^{\kappa\lambda} &= \tilde{E}^{\kappa\lambda\mu\nu}\tilde{p}_{\mu\nu} \\ \tilde{p}_{\kappa\lambda} &= \tilde{E}_{\kappa\lambda\mu\nu}\tilde{p}^{\mu\nu}. \end{aligned} \tag{11}$$

From Projective to Affine Space

Consider the projective space of the previous section, and in it single out a plane π, called the *improper plane* (but any other preferred plane would do equally well) and represented by a_λ. Thus there is singled out a point on every line not in π and a line on every plane not coinciding with π. It will be convenient to call points and lines in π, as well as planes whose support is a_λ, *improper* and all others *proper*.

Since geometry of projective three-space with an improper plane is isomorphic with affine geometry, lines intersecting in a point of π will be called *parallel*, and planes intersecting in a line of π will likewise be *parallel*. Now, since every line or plane not contained in π has a preferred point or line, it is possible to distinguish between the two directions along a line and the two sides of a plane.

It will be convenient to introduce a normalization by assigning to the improper plane—or "plane at infinity"—an absolute covector a_λ. The common way of doing this is to select for a_λ the coordinate plane (1,0,0,0) and to consider only those transformations of the coordinate system in which a_λ preserves these coordinates. Then to every contravariant vector x^κ can be assigned a *weight* x^0,

$$x^\kappa a_\kappa = x^0. \tag{12}$$

A system of four homogeneous coordinates in affine space is set up when an origin and three unit points are chosen. These form a coordinate tetrahedron. In the plane at infinity there is a system of three homogeneous coordinates forming a coordinate triangle. Therefore a point in the plane at infinity can be represented either by four coördinates of affine space x^κ or by three coordinates v^i in the plane at infinity. A point not in the plane at infinity has only one coordinate expression x^κ. Obviously there is a linear relation between the coordinates x^κ and v^i in the plane at infinity, such that $v^\kappa = x^\kappa(v^i)$, which may be written

$$x^\kappa = B^\kappa_i v^i. \tag{13}$$

In places where no ambiguity can arise, we can replace v^i by x^i, in this way indicating that both x^κ and x^i are different ways of expressing the same geometrical entity.

If the unit points of the coordinate tetrahedron and the coordinate triangle in the plane at infinity are made to correspond, then the matrix B^κ_i is written

$$B^\kappa_i = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}. \tag{14}$$

A plane which does not coincide with the plane at infinity intersects the plane at infinity in a line. All parallel planes intersect the plane at infinity in the same line. A free plane may thus be associated with this line in the plane at infinity, or

$${}^+w_i = B^\lambda_i w_\lambda. \tag{15}$$

Affine Figures Represented by Contravariant Tensors of Valence One

Affine figures are those gemometrical figures which are invariant under affine transformations. Consider first such figures represented by contravariant tensors.

A *weighted point* is represented by a contravariant tensor x^s which transforms by the law

$$x^{s'} = A^{s'}_{s} x^s, \qquad \Delta = |A^{s'}_{s}| \neq 0. \tag{16}$$

If the point lies in the improper plane,

$$x^s a_s = x^0 = 0, \tag{17}$$

and the weight x^0 must be zero. The *support* of a point is an unpolarized, unnormalized point $\bar{x}^s$, and we can distinguish between points of different weight and the same support by the weight $|x^0|$ and the polarization or sign of the weight. A proper point is represented by

$$x^s = (x^0, x^1, x^2, x^3), \qquad x^0 \neq 0. \tag{18}$$

If two weighted points x^s and y^s have the same support,

$$x^s = \tau y^s. \tag{19}$$

Two weighted points are identical if, and only if, $\tau = 1$.

An improper point has $x^0 = 0$ in (18). Its support is the unpolarized and unnormalized direction $\bar{x}^i$. Another way of considering the improper point is to look at the tensor as a *free affine vector*. This is a free direction with polarization and normalization. The coordinates of the free vector v^i may be defined as the difference of two proper points of equal weight, x^s and y^s, or

$$v^s = x^s - y^s, \qquad v^s = B^s_i v^i \qquad \text{if } x^0 = y^0. \tag{20}$$

Since we can distinguish between the order in which these points are taken, an inner affine polarization is defined. There is an inner affine normalization factor λ by which two free affine vectors v^i and w^i, having the same support, can be compared:

$$v^i = \lambda w^i. \tag{21}$$

Multiplication of a proper or improper point by a scalar σ gives another point of the same support, with normalization factor $|\sigma|$ times the original one. If σ is negative, the polarization is also reversed.

Addition of proper points and free affine vectors satisfies the commutative, associative, and distributive laws. If a set of proper points all have the same polarization, the construction is the familiar one of finding the center of gravity of the weighted points and replacing them by a point of weight equal to the sum of the component weights at the center of gravity. When points of both polarizations are included, the construction has no particular mechanical analogue. The addition of free affine vectors may be effected geometrically by the familiar parallelogram construction, and always gives another free affine vector or the null point, $x^s \equiv 0$.

Affine Figures Represented by Covariant Tensors of Valence One

A plane is represented by a covariant tensor w_λ which transforms by the law

$$w_{\lambda'} = A^{\lambda}_{\lambda'} w_\lambda \qquad \Delta = |A^{\lambda}_{\lambda'}| \neq 0. \tag{22}$$

If the plane is an improper one,

$$w_\lambda = w_0 : 0 : 0 : 0. \tag{23}$$

Then the support of the plane is an unpolarized and unnormalized plane and we can distinguish between planes of different normalizations and the same support. If $w_0 = 0$, the plane passes through the origin of coordinates.

A proper plane is represented by

$$w_\lambda = w_0 : w_1 : w_2 : w_3 \qquad w_0 \neq 0. \tag{24}$$

The improper plane has $w_i \equiv 0$. Two proper planes having the same support can be compared by

$$w_\lambda = \tau v_\lambda , \tag{25}$$

where $|\tau|$ is the inner affine normalizing factor of the two proper planes. The proper plane has an outer polarization (the two sides may be distinguished), since if a_λ is an improper plane,

$$w_{[\lambda} a_{\kappa]} \neq a_{[\lambda} w_{\kappa]} . \tag{26}$$

Two improper planes (with necessarily the same support $\tilde{a}_\lambda$) can be compared by

$$w_0 = \tau v_0 , \qquad w_i = v_i = 0, \tag{27}$$

where $|\tau|$ is the inner affine normalization factor of the two improper planes and may be thought of as a weighted affine area.

The free affine plane was introduced in an earlier paragraph. Its tensor representation is ${}^{+}w_i$. The support of this figure is an unpolarized and unnormalized free plane $\tilde{w}_i$. In addition, it has polarization and normalization. As with the proper plane, there is an outer affine polarization by which the two sides of the plane may be distinguished. Likewise there is an inner affine normalization factor which may be thought of as a weighted affine area.

Multiplication of a proper, improper, or free plane by a scalar σ multiplies the weighted affine area by a scalar $|\sigma|$. Also, if σ is negative, the polarization is reversed.

Addition of proper, improper, and free planes satisfies the familiar commutative, associative, and distributive laws.

Combination of Tensors of Valence One

One way of discovering bivalent geometric figures with interesting properties is to take pairs of the figures represented by tensors of valence one and combine them. There is quite a variety of combinations. With two contravariant tensors of valence one, the simplest combination is obtained by taking the *alternating product* of the two tensors:

$$p^{\omega} = x^{[\kappa} y^{\lambda]} .$$

Similarly with covariant tensors of valence one,

$$q_{\alpha\lambda} = u_{[\alpha}v_{\lambda]}.$$

If one tensor is contravariant and the other covariant, the simplest nontrivial combination is found by taking a *direct product*,

$$p^{\kappa}_{.\lambda} = x^{\kappa}u_{\lambda}.$$

Besides multiplication, one may use addition to produce still other figures having properties that are different from those of the original ones.

Figures in Affine Space Represented by Contravariant Alternating Tensors of Valence Two

In this section, contravariant alternating tensors of valence two obtained in the above manner will be considered.

It is convenient to introduce the following notation:

A figure defined in terms of two contravariant figures of valence one will be said to be

proper if made up of two proper elements,

semi-proper if made up of one proper element and one improper element,

improper if made up of two improper elements.

Consider those figures whose tensor representation is an alternating contravariant tensor of valence two that satisfies Plücker's relation.[1] Their tensor representation is a "bivector."

The proper figure[2] determined by a pair of distinct proper points x^{κ} and y^{κ} is defined by the equation,

$$p^{\alpha\lambda} = x^{[\alpha}y^{\lambda]}, \qquad p^{[\alpha\lambda}p^{\mu\nu]} = 0. \tag{28}$$

This equation represents a line with polarization and normalization. The support of the figure[2] is the unpolarized and unnormalized line,

$$\bar{p}^{\alpha\lambda} = -\bar{p}^{\lambda\alpha}, \qquad p^{[\alpha\lambda}p^{\mu\nu]} = 0. \tag{29}$$

The restriction on this figure[2] is that $p^{0i} \neq 0$ unless $p^{\alpha\lambda}$ degenerates into the null figure[1], $p^{\alpha\lambda} = 0$. For if $p^{0i} = 0$ then either, one of the points in terms of which it is defined is improper, or they are coincident; and both of these cases are disallowed by the above definition of the figure of two proper points.[2]

Inner polarization (order of points on the support or direction along it) is defined, since

$$p^{\alpha\lambda} \neq p^{\lambda\alpha} \quad \text{but} \quad p^{\alpha\lambda} = -p^{\lambda\alpha}. \tag{30}$$

To find the meaning of the normalization of the figure of two proper points[2] we redefine $p^{\alpha\lambda}$ in terms of another pair of points lying on the support,

$$p^{\alpha\lambda} = X^{[\alpha}Y^{\lambda]}. \tag{31}$$

The points X^κ and Y^κ may be written in terms of x^κ and y^κ,

$$X^\kappa = ax^\kappa + by^\kappa$$
$$Y^\kappa = cx^\kappa + dy^\kappa. \tag{32}$$

Substitution of (32) in (31) gives,

$$p^{\kappa\lambda} = acx^{[\kappa}x^{\lambda]} + adx^{[\kappa}y^{\lambda]} + bcy^{[\kappa}x^{\lambda]} + bdy^{[\kappa}y^{\lambda]}$$
$$= (ad - bc)x^{[\kappa}y^{\lambda]}. \tag{33}$$

If $ad - bc = 1$, the figure[2] defined by x^κ and y^κ and that defined by X^κ and Y^κ are identical. Any pair of points on the support for which this is true are said to be of equal *separation* and determine the same proper figure[2]. The separations of pairs of points on different lines cannot be compared. If the pairs of points are of different separation ($ad - bc = \lambda \neq 1$) but of the same polarization

$$p^{\kappa\lambda} = X^{[\kappa}Y^{\lambda]} = \lambda x^{[\kappa}y^{\lambda]} \tag{34}$$

and λ is the inner affine normalization factor of $p^{\kappa\lambda}$. If $\lambda = 0$ the null figure[1] is obtained. All null figures of two proper points are said to be equivalent.

Hence the figure of two proper points[2] is unchanged if it is defined by any ordered pair of proper points on its support. If the separation of the points on the support is changed, the figure[2] is multiplied by a scalar. Multiplication by a scalar reverses the polarization and changes the normalization if the scalar is negative.

From the geometric point of view, another way of obtaining a geometrical figure is to consider the combination of a proper point x^κ and a free vector v^i. This figure[3] is defined by the equation

$$p^{\kappa\lambda} = x^{[\kappa}v^{|i|}B_i^{\lambda]}, \qquad p^{[\kappa\lambda}p^{\mu\nu]} = 0. \tag{35}$$

As with the definition of the figure of two proper points,[2] the tensor $p^{\kappa\lambda}$ has inner polarization and normalization. Its polarization is the same as that of the free vector v^i. It is independent of the proper point x^κ as long as x^κ remains on the support. The inner affine normalization of $p^{\kappa\lambda}$ and v^i are identical.

The tensor representations of the figures of two proper points[2] and of a proper point and a free vector[3] are identical. Both have line support and inner polarization and normalization. There may be some question as to whether they should be considered as distinct geometrical figures. From this point of view, they should since they are defined in terms of different geometrical elements. The analogous covariant figures are utterly distinct. If the method were not to distinguish these figures, several other more important distinctions might be lost in the subsequent development.

The remaining figure[4] of this type is determined by two free affine vectors as follows,

$$p^{ij} = v^{[i}w^{j]}, \qquad p^{[ij}p^{kl]} = 0. \tag{36}$$

This represents a polarized and normalized free plane, whose support is the plane direction $\tilde{p}^{ij}$. The polarization is an inner one (sense of rotation) in the free plane. This is determined by the order in which the free affine vectors are combined since,

$$p^{ij} = -p^{ji}. \tag{37}$$

The same tensor representation is obtained if any other pair of free affine vectors, contained in the support $\tilde{p}^{ij}$ are chosen, subjected to conditions similar to those of Eq. (33) with $ad - bc = 1$. If we multiply either v^i or w^i by a constant σ, then p^{ij} is multiplied by σ. The normalization of the figure of two free vectors[4] is then determined by the affine area of the parallelogram defined by v^i and w^i.

Proper, semi-proper, and improper figures defined in terms of points[1] whose supports intersect satisfy the usual commutative, associative, and distributive laws of addition.

Now take up figure[5] whose tensor representation is an alternating contravariant tensor of valence two, that does not in general satisfy Plücker's relation. The general sum of figures defined in terms of points[1] is a figure of the foregoing type.[5] The generalized sum of the figures defined in terms of two proper points[2] and two free vectors[4] is a proper figure of this type.[6] The generalized sum of semi-proper and improper figures of points[1] is a semi-proper figure of this type.[7] In the special case in which $p^{[\alpha\lambda}p^{\mu\nu]} = 0$ while $p^{oi} \neq 0$ the proper and semi-proper figures[5] degenerate into proper and semi-proper figures of points.[1] In the second special case in which $p^{oi} \equiv 0$, the proper and semi-proper figures[5] degenerate into figures of two free vectors.[4]

A figure represented by $p^{\alpha\lambda}$ can be decomposed with respect to a given proper point x^α into a proper figure of two points[2] $q^{\alpha\lambda}$ whose support contains x^α and an improper figure of two free vectors[4] r^{kl}. Then

$$p^{\alpha\lambda} = q^{\alpha\lambda} + r^{kl}B^\alpha_k B^\lambda_l\,, \tag{38}$$

where $q^{\alpha\lambda} = x^{[\alpha}y^{\lambda]}$ and $r^{kl} = u^{[k}v^{l]}$. Since the point x^α is given, y^α can be found as follows,

$$\begin{aligned} p^{oi} &= x^o y^i - y^o x^i \\ y^i &= \frac{p^{oi} + y^o x^i}{x^o} \end{aligned} \tag{39}$$

and if the weight of y^α is arbitrarily chosen equal to that of x^α

$$\begin{aligned} y^i &= \frac{p^{oi}}{x^o} + x^i \\ y^o &= x^o \end{aligned} \tag{40}$$

or since

$$p^{oo} = 0$$

$$y^\alpha = \frac{p^{o\alpha}}{x^o} + x^\alpha. \tag{41}$$

Therefore one component[2] of the decomposition with respect to x^κ is

$$q^{\alpha\lambda} = x^{[\alpha} y^{\lambda]} = \frac{x^{[\alpha} p^{|\sigma|\lambda]}}{x^\sigma} \tag{42}$$

The other component[4] of the decomposition with respect to x^κ is

$$B^\alpha_k B^\lambda_l r^{kl} = p^{\alpha\lambda} - q^{\alpha\lambda} = p^{\alpha\lambda} - \frac{x^{[\alpha} p^{|\sigma|\lambda]}}{x^\sigma} \tag{43}$$

This proper figure[6] defines a field of such pairs of a proper and an improper rhabdel, one pair of which is associated with every proper point. The semi-improper figure[7] can be decomposed in a strictly similar manner.

Sums of proper and semi-proper figures[5] of this type are again proper or semi-proper figures[5] of the same type. They are added when their decompositions with respect to any proper point are added. When a figure of this type[5] is multiplied by a scalar the affine normalization of the proper, semi-proper, or improper figures of points[1] are multiplied by a constant. Thus, there is no new geometric construction which need be introduced here and *only one method of addition*. Addition of figures of this type[5] satisfies the commutative, associative, and distributive laws.

Figures in Affine Space Represented by Covariant Alternating Tensors of Valence Two

In this section the covariant alternating tensors of valence two obtained in a similar manner will be considered. Since the principle of duality is no longer valid in affine space, there is no reason to expect parallelism between this section and the previous one. There are three covariant figures of valence one in affine space, the *proper plane*, the *improper plane*, and the *free plane*. Taking alternating products of pairs of these there are six possible covariant figures of valence two. That formed by two improper planes is identically zero, while those formed by one improper plane and one proper plane or free plane are not distinct. This leaves four covariant figures to be discussed.

It is convenient to introduce the following notation:

A figure defined in terms of two covariant figures of valence one will be said to be

proper if made up of two proper elements
free if made up of two free elements
semi-attached if made up of one proper element and one free element
semi-free if made up of one improper element and one free element.

Consider figures[8] whose tensor representation is an alternating covariant tensor of valence two that satisfies Plücker's relation.

A *proper* figure of two planes[9] is determined by a pair of distinct, non-parallel, proper planes u_λ and v_λ and is defined by the equation

$$p_{\alpha\lambda} = u_{[\alpha} v_{\lambda]}\,, \qquad p_{[\alpha\lambda} p_{\mu\nu]} = 0. \tag{44}$$

The support of the figure[9] is the unpolarized and unnormalized line,

$$\bar{p}_{\kappa\lambda} = -\bar{p}_{\lambda\kappa}\,, \qquad \bar{p}_{[\kappa\lambda}\bar{p}_{\mu\nu]} = 0. \tag{45}$$

The restrictions on the proper figure of two planes[9] are first that $p_{ij} \not\equiv 0$ unless $p_{\kappa\lambda}$ degenerates into the null figure[8] $p_{\kappa\lambda} \equiv 0$. For if $p_{ij} \equiv 0$ then the planes are parallel, improper, or coincident and these cases are all disallowed by the above definition. The second restriction is that $p_{i_0} \not\equiv 0$, for if $p_{i_0} \equiv 0$, then either both planes pass through the origin and are coincident or both planes are free. These situations are also disallowed by the definition.

Outer polarization (order of planes intersecting in the support or direction about it) is defined, since

$$p_{\kappa\lambda} \not\equiv p_{\lambda\kappa} \quad \text{but} \quad p_{\kappa\lambda} = -p_{\lambda\kappa}\,. \tag{46}$$

To find the meaning of the normalization of the proper figure of two planes[9] we redefine $p_{\kappa\lambda}$ in terms of another pair of distinct, non-parallel, proper planes which intersect in the support $\bar{p}_{\kappa\lambda}$,

$$p_{\kappa\lambda} = U_{[\kappa}V_{\lambda]}\,. \tag{47}$$

The conditions that a plane U_κ lie on a line determined by u_κ and v_κ are

$$\begin{aligned} U_\kappa &= au_\kappa + bv_\kappa \\ V_\kappa &= cu_\kappa + dv_\kappa\,. \end{aligned} \tag{48}$$

Substitution of (48) in (47) gives,

$$\begin{aligned} p_{\kappa\lambda} &= acu_{[\kappa}u_{\lambda]} + adu_{[\kappa}v_{\lambda]} + bcv_{[\kappa}u_{\lambda]} + bdv_{[\kappa}v_{\lambda]} \\ &= (ad - bc)u_{[\kappa}v_{\lambda]}\,. \end{aligned} \tag{49}$$

If $ad - bc = 1$, the proper figure[9] defined by u_κ and v_κ and that defined by U_κ and V_κ are identical. Any pair of planes for which this is true are said to be of equal opening and determine the same proper figure.[9] The opening of pairs of planes intersecting in different lines cannot be compared. If the pairs of planes are of different openings ($ad - bc = \lambda \neq 1$) but of the same polarization

$$p_{\kappa\lambda} = U_{[\kappa}V_{\lambda]} = \lambda u_{[\kappa}v_{\lambda]} \tag{50}$$

and λ is the outer normalization factor of $p_{\kappa\lambda}$. If $\lambda = 0$ the null figure[8] is obtained. All null figures of planes[8] are said to be equivalent.

Hence a proper figure of two planes[9] is unchanged if it is defined by any ordered pair of planes of equal opening intersecting in its support. If the opening is changed, the proper figure of two planes[9] is multiplied by a scalar. Multiplication by a scalar reverses the polarization and changes the normalization, if the scalar is negative.

A figure[10] determined by a pair of distinct free planes is represented by

$$p_{kl} = {}^{+}u_{[k}{}^{+}v_{l]}\,, \qquad p_{[ij}p_{kl]} = 0. \tag{51}$$

This represents a polarized and normalized free direction. The support of the free figure[10] is the unpolarized and unnormalized free direction

$$\tilde{p}_{kl} = -\tilde{p}_{lk}\,, \qquad \tilde{p}_{[ij}\tilde{p}_{kl]} = 0. \tag{52}$$

Outer polarization (order of free planes intersecting in the support, or directions about support) is defined, since

$$p_{kl} \neq p_{lk} \quad \text{but} \quad p_{kl} = -p_{lk}\,. \tag{53}$$

A discussion nearly identical with that given for the proper figure of two planes[9] shows immediately that the polarization is outer and, in general, corresponds to the opening between the free planes. Then the same tensor representation is obtained if the free figure of two planes[10] is defined in terms of any ordered pair of planes of equal opening intersecting in the support. If the normalization of one of the free planes is multiplied by a scalar, then the free figure[10] is multiplied by a scalar. If the free figure[10] is multiplied by a scalar, the opening of the planes is multiplied by a scalar and if the scalar is negative, the polarization is reversed.

A *semi-free* figure[11] is determined by one improper plane and one free plane and its equation is

$$p_{oi} = u_o{}^{+}v_i = -p_{io}\,, \tag{54}$$

where u_o is an improper plane and v_i is a free plane. This equation represents a free plane with polarization and normalization. The support of the semi-free figure[11] is the unpolarized and unnormalized free plane,

$$\tilde{p}_{oi} = -\tilde{p}_{io}\,. \tag{55}$$

Outer polarization (method of distinguishing between the sides of the support) is defined, since

$$p_{oi} \neq p_{io} \quad \text{but} \quad p_{oi} = -p_{io}\,. \tag{56}$$

The outer normalization is defined as the product of the affine normalization of the improper plane and the free plane. The semi-free figure[11] is multiplied by a scalar when the normalization of either plane is multiplied by a scalar. When a semi-free figure[11] is multiplied by a negative scalar its normalization and polarization are both unchanged.

A semi-attached figure[12] is determined by a proper plane u_λ and a free plane ${}^{+}v_l$ which are not parallel or coincident. Its defining equation cannot be written in a single tensor equation, though it may be expressed by the following pair of equations

$$\begin{aligned} p_{kl} &= u_{[k}{}^{+}v_{l]}\,, \qquad p_{[kl}p_{mn]} = 0 \\ p_{oi} &= u_o v_i = -p_{io}\,. \end{aligned} \tag{57}$$

This is a different type of figure from those previously considered. It is a pair of a free[10] and a semi-free[11] figure. This figure will not be discussed further in this paper.

Proper, free, and semi-free figures of planes[8] whose supports intersect satisfy the usual commutative, associative and distributive laws.

A figure[13] whose tensor representation is an alternating covariant tensor of valence two, that does not in general satisfy Plücker's relation, will now be taken up. The general sum of figures of planes[8] is of this type.[13]

The generalized sum of proper, free, and semi-free figures of planes[8] is a proper figure,[14] $p_{\kappa\lambda}$. In the special cases in which $p_{[\kappa\lambda}p_{\mu\nu]} = 0$ while neither $p_{ij} \equiv 0$ nor $p_{oi} \equiv 0$, the proper figures[14] degenerate into proper figures of planes.[9] In the second special case in which $p_{oi} \equiv 0$ the proper figure[14] degenerates into a free figure of planes.[10] In the third special case in which $p_{ij} \equiv 0$ and $p_{oi} \not\equiv 0$, this figure[14] degenerates into a semi-free figure of planes.[11]

A proper figure[14] of this type can be decomposed into figures of planes[8] in various ways.

The first method of decomposition is with respect to a proper point. Given a proper point x^κ the figure[13] can be broken up into a proper figure of planes[9] whose support contains x^κ and a semi-free figure of planes.[11] For any figure[13] $p_{\kappa\lambda}$ there always exists a pair of planes U_κ and V_κ such that

$$p_{kl} = {}^+U_{[k}{}^+V_{l]} \tag{58}$$

and

$$U_o = \frac{x^k U_k}{x^o}, \qquad V_o = \frac{x^k V_k}{x^o}. \tag{59}$$

The proper[9] component whose support contains x^κ is then

$$q_{\kappa\lambda} = U_{[\kappa}V_{\lambda]}. \tag{60}$$

The semi-free component which then remains is

$$\begin{aligned} r_{oi} &= p_{oi} - U_o V_i + V_o U_i \\ &= p_{oi} - \frac{x^k U_k}{x^o} V_i + \frac{x^k V_k}{x^o} U_i. \end{aligned} \tag{61}$$

The proper planes U_κ and V_κ are not unique but their support and affine angle are given by x^κ and $p_{\kappa\lambda}$.[9] Therefore a proper[9] and a semi-free[11] figure of planes are uniquely defined with respect to the point x^κ.

The second method of decomposition is dual to the decomposition described in the previous chapter. A proper figure[14] can be decomposed with respect to a given proper plane U_λ into a proper figure of planes[9] $q_{\kappa\lambda}$ whose support contains U_λ and a free figure of planes[10] r_{kl}. Let

$$\begin{aligned} p_{k\lambda} &= q_{k\lambda} + r_{kl}B^l_\lambda \\ p_{o\lambda} &= q_{o\lambda} \end{aligned} \tag{62}$$

where

$$q_{\kappa\lambda} = U_{[\kappa}V_{\lambda]} \quad \text{and} \quad r_{kl} = {}^+U_{[k}{}^+V_{l]}.$$

Since the proper plane U_λ is given, V_λ can be found as follows,

$$p_{oi} = U_o V_i - V_o U_i$$

$$V_i = \frac{p_{oi} + V_o U_i}{U_o} \tag{63}$$

and if the weight of V_λ is arbitrarily chosen equal to that of U_λ (this can be done for the weight of U_λ is also arbitrary)

$$V_s = \frac{p_{os}}{U_o} + U_s . \tag{64}$$

Therefore the proper[9] component of the decomposition of the figure[13] with respect to U_s is

$$q_{s\lambda} = U_{[s} V_{\lambda]} = \frac{U_{[s} p_{|o|\lambda]}}{U_o} . \tag{65}$$

The free[10] component of the decomposition of the figure[13] with respect to U_λ is

$$r_{kl} = p_{kl} - q_{kl} = p_{kl} - \frac{{}^{+}U_{[k} p_{|o|l]}}{U_o} . \tag{66}$$

Then with each proper plane U_λ is uniquely associated a proper figure of planes[9] support lies in U_λ, and a free figure of planes.[10]

The proper figure of planes[14] can also be decomposed into two free figures,

$$p_{kl} = q_{kl}$$

$$p_{ol} = r_{ol} . \tag{67}$$

This gives a decomposition into a free and a semi-free[11] figure of planes. Through every proper point passes a representation of this decomposition of this figure.[13]

Correspondences between Figures in Affine Space Represented by Contravariant and Covariant Alternating Tensors of Valence Two

In projective space a fundamental pseudotensor was introduced by which contravariant and covariant tensor representations of a line were associated. By means of the fundamental pseudotensor, an association was then set up between all contravariant and covariant alternating tensors of valence two. In affine space, without a perfect principle of duality, it is not possible to obtain a complete 1 : 1 correspondence of all contravariant and covariant alternating tensors of valence two. However, a fundamental tensor can be introduced which permits a very useful partial correspondence. This will be called the *E*-tensor. Since it is introduced into affine space for the purpose of associating contravariant and covariant alternating tensors of valence two, it must obviously be of valence four, have both a contravariant and a covariant representa-

tion, and be alternating in any pair of indices. For definiteness, it is convenient to define the E-tensor of affine space in the following manner,

$$
\begin{aligned}
E^{\kappa\lambda\mu\nu} &= 0 \text{ if any pair of indices are equal} \\
&= 1 \quad \text{if } \kappa\lambda\mu\nu \text{ is an even permutation of } 0, 1, 2, 3. \qquad (68) \\
&= -1 \text{ if } \kappa\lambda\mu\nu \text{ is an odd permutation of } 0, 1, 2, 3. \\
E_{\kappa\lambda\mu\nu} &= 0 \quad \text{if any pair of indices are equal} \\
&= +1 \quad \text{if } \kappa\lambda\mu\nu \text{ is an even permutation of } 0, 1, 2, 3. \qquad (69) \\
&= -1 \quad \text{if } \kappa\lambda\mu\nu \text{ is an odd permutation of } 0, 1, 2, 3.
\end{aligned}
$$

Consider a proper figure of points[2] $p^{\kappa\lambda}$ and proper figures of planes[9] $p_{\kappa\lambda}$. Let,

$$
p^{\kappa\lambda} = E^{\kappa\lambda\mu\nu}p_{\mu\nu}, \qquad p_{\kappa\lambda} = E_{\kappa\lambda\mu\nu}p^{\mu\nu}. \tag{70}
$$

This equation shows first that these figures have the same support, the line $\tilde{p}^{\kappa\lambda}$ or $\tilde{p}_{\kappa\lambda}$. If the polarization of the figure of proper points[2] is reversed, that of the corresponding figure of proper planes[9] is also reversed. Thus inner and outer polarization of a line are associated and a screw-sense is introduced into the space. Either a right or left hand screw sense may be chosen but once it is arbitrarily fixed it cannot be lightly reversed. If the figure of proper points[2] is multiplied by a scalar, then figure of proper planes[9] is also multiplied by a scalar. Thus a 1:1 correspondence is set up between inner and outer affine normalization of a line. This definition could obviously be altered if a more convenient convention presented itself. Sums of proper figures of planes[8] and of points whose supports intersect (and hence are again proper) also correspond.

Consider now an improper figure of points[4] p^{kl} and a semi-free figure of planes[11] p^{oi}. Let,

$$
p^{ij} = E^{ijok}p_{ok}, \qquad p_{oi} = E_{oijk}p^{jk}. \tag{71}
$$

This equation shows that these figures have the same plane support. For any line in the support of the one coincides with a line in the support of the other. Coincidence of two such distinct lines fixes the plane direction which is the support of both figures. Inner and outer polarization of a plane are associated in such a way that the screw sense introduced arbitrarily above is continued. Also inner and outer affine normalizations are associated in a 1:1 way, the exact relation between the two normalizations being fixed arbitrarily. Sums of improper figures of points[4] are always improper and sums of semi-free figures of plates are always semi-free figures of plates. Improper figures of points and semi-free figures of plates always correspond.

The other figures of points[1] and planes[8] cannot be put in a 1:1 correspondence in this manner.

The tensors representing sums of figures of points[5] and planes[13] are rather abstract ways of considering systems of figures of points[1] and planes[8]. When we compare them what we want to compare is actually their decomposition with respect to some proper point.

The proper figure of the sum of figures of points[6] can be decomposed with respect to a given point into a proper[2] and an improper[4] figure of points. The proper figure of the sum of figures of planes[14] can be decomposed with respect to the same proper point into a proper[9] and a semi-free figure of planes.[11] The figures of the decomposition have the same support and there is a unique 1:1 correspondence between them. Therefore when proper figures of sums of figures of points[5] and planes[13] are added their decompositions with respect to any given proper point are added. This is the meaningful way of treating them.

It is also possible to consider one of the other decompositions of the proper figure of sums of figures of planes.[14] These decompositions have no 1:1 correspondence with figures of points[1] and a comparison of additions of non-corresponding figures as such is meaningless.

Figures in Affine Space Represented by Mixed Tensors of Valence Two and Rank One

Mixed tensors of valence two and rank one can be obtained by taking the direct product of the elements of affine space. There were introduced two contravariant figures, the weighted point, and the free vector, and three covariant ones, the weighted plane, the improper plane, and the free plane. Taking direct products of one of the contravariant and one of the covariant figures six tensors of rank one are obtained. The properties of these will be taken up in this section, together with their sums which are tensors of rank four.

It is convenient to extend the earlier notation in the following manner:

A figure defined in terms of one contravariant and one covariant figure of valence one will be said to be *proper* if made up of two proper elements, *semi-proper* if made up of one proper contravariant element and one improper covariant element, *semi-improper* if made up of one improper contravariant element and one proper covariant element, *semi-attached* if made up of one proper contravariant element and one free covariant element, *free* if made up of two free elements, *improper* if made up of two improper elements.

Consider figures[15] whose tensor representation is a mixed tensor of valence two and rank one.

The first figure[16] is determined by a proper point x^s and a proper plane u_λ which are not coincident. Its tensor representation has the equation

$$p^s_{\cdot\lambda} = x^s u_\lambda . \tag{72}$$

This tensor is obviously of rank one since its matrix is the product of two matrices of rank one. The geometrical representation is a definite normalized point and plane the product of whose normalizations is given. The support is an unpolarized and unnormalized pair of a point and a plane whose expression is

$$p^s_{\cdot\lambda} . \tag{73}$$

[1] This number refers to the names developed in the Appendix.

The proper point in terms of which the figure of a proper point and plane[16] is defined can be found as follows if p^{ϵ}_{λ} is given,

$$x^{\epsilon} = \tau p^{\epsilon}_{\cdot o} \qquad \tau \text{ arbitrary scalar} \tag{74a}$$

while the proper plane may be found by

$$u_{\lambda} = \frac{1}{\tau p^{o}_{\cdot o}} p^{o}_{\cdot \lambda} . \tag{74b}$$

A polarization may be defined by the order in which the point and plane are taken, since

$$p^{\epsilon}_{\cdot \lambda} \neq p^{\cdot \epsilon}_{\lambda} . \tag{75}$$

Reversal of polarization interchanges rows and columns of the matrix of the tensor representation.

Any two figures of points and planes[15] whose support is the same point and plane may be compared by an affine normalizing factor λ since

$$p^{\epsilon}_{\cdot \lambda} = \lambda q^{\epsilon}_{\cdot \lambda} . \tag{76}$$

The normalization factor is the product of those of the point and plane.

This figure is especially interesting because it has a type of support we have not met before.

A semi-proper figure[17] is defined by a proper point x^{ϵ} and an improper plane u_o,

$$p^{\epsilon}_{\cdot o} = x^{\epsilon} u_o . \tag{77}$$

The support of the figure is the unpolarized and unnormalized point $\bar{x}^{\epsilon}$ or

$$\bar{p}^{\epsilon}_{\cdot o} . \tag{78}$$

A polarization may be given by the order in which point and plane are taken since

$$p^{\epsilon}_{\cdot o} \neq p^{\cdot \epsilon}_{o} . \tag{79}$$

Reversal of polarization changes the matrix of the tensor representation from a row-matrix to a column matrix or vice versa.

The affine normalization can be defined for two figures having the same support since then

$$p^{\epsilon}_{\cdot o} = \lambda q^{\epsilon}_{\cdot o} . \tag{80}$$

The affine normalization factor is the product of the weight of the proper point and the improper plane. They may vary on the support in such a way that their product remains constant.

A *semi-improper* figure[18] is defined by an improper point x^{i} and a proper plane u_{λ}. Its tensor equation is

$$p^{i}_{\cdot \lambda} = x^{i} u_{\lambda} . \tag{81}$$

The support of the figure is an unpolarized and unnormalized proper plane and a free direction

$$\bar{p}^{i}_{.\lambda}\,, \tag{82}$$

of which the plane is the support of u_λ and the direction that of x^i.

A polarization of this semi-improper figure[18] may be determined by the order in which point and plane are taken. Thus

$$p^{i}_{.\lambda} \neq p_{\lambda}^{\cdot i}, \tag{83}$$

and reversal of polarization interchanges rows and columns of the matrix associated with the figure.

The semi-improper figure of a point and a plane[18] has affine normalization defined for two figures having the same support since then

$$p^{i}_{.\lambda} = \lambda q^{i}_{.\lambda}\,. \tag{84}$$

The semi-improper figure of a point and a plane[18] is unchanged if the normalization of the proper plane and improper point are varied in such a way that their product remains constant.

A *semi-attached* figure[19] is defined as the figure determined by a proper point x^s and a free plane ${}^{+}u_i$. Its equation is

$$p^{s}_{.i} = x^{s+}u_i\,. \tag{85}$$

The support of the figure is a point and a plane direction

$$\bar{p}^{s}_{.i} \tag{86}$$

A polarization of the semi-attached figure[19] may be determined by the order in which proper point and free plane are taken since

$$p^{s}_{.i} \neq p_{i}^{\cdot s}. \tag{87}$$

Reversal of polarization interchanges rows and columns of the coordinate matrix.

This semi-attached figure[19] has affine normalization defined for two figures having the same support, since then

$$p^{s}_{.i} = \lambda q^{s}_{.i}\,. \tag{88}$$

It is unchanged if the normalizations of proper point and free plane are varied in such a way that their product remains constant.

A *free* figure[20] is defined by a free vector (or an improper point) and a free plane, by the equation,

$$p^{s}_{.i} = x^{s+}u_i\,. \tag{89}$$

Its support is a free direction and a free plane direction

$$\bar{p}^{s}_{.i}\,. \tag{90}$$

A polarization may be defined by the order in which point and plane are taken since

$$p^{\epsilon}_{\cdot i} \neq p^{\cdot \epsilon}_{i}, \tag{91}$$

and when polarization is reversed rows and columns are interchanged.

The affine normalization of this free figure[20] is defined for two figures having the same support by

$$p^{\epsilon}_{\cdot i} = \lambda q^{\epsilon}_{\cdot i}\,. \tag{92}$$

It is unchanged if the affine normalization of the free vector and the free plane vary in such a way that their product remains constant.

An improper figure[21] is defined by an improper point x^i and an improper plane u_o. Its equation is

$$p^{i}_{\cdot o} = x^i u_o\,. \tag{93}$$

Its support is an unpolarized and unnormalized direction, that of x^i, or

$$\bar{p}^{i}_{\cdot o}\,. \tag{94}$$

The order in which the improper point and the improper plane are taken, may be taken as a polarization since

$$p^{i}_{\cdot o} \neq p^{\cdot i}_{o}. \tag{95}$$

By this definition, the polarization is reversed when rows and columns of the coordinate matrix are interchanged. For two such improper figures[21] having the same support the affine normalization is defined by the equation

$$p^{i}_{\cdot o} = \lambda q^{i}_{\cdot o}\,. \tag{96}$$

This figure[21] is unchanged if the affine normalization of the improper point and of the free plane vary in such a way that their product remains constant.

Figures of points and planes[15] whose supports have a common member, satisfy the ordinary commutative, associative, and distributive laws of addition.

Consider figures[22] whose tensor representation is a mixed tensor of valence two and rank four, the general sum of figures of points and planes.[15]

The general sum of proper figures of points and planes[15] is a *proper figure.*[23] This proper figure[23] can be decomposed into three proper figures of points and planes[15] and one semi-proper figure of a point and a plane[17] with respect to any three arbitrary non-parallel planes. It can also be decomposed into four proper figures of points and planes[16] with respect to any four distinct proper points or any four non-parallel planes. In the special case in which

$$\begin{aligned} p^{o}_{\cdot\lambda} &= k_1 p^{1}_{\cdot\lambda} = k_2 p^{2}_{\cdot\lambda} = k_3 p^{3}_{\cdot\lambda} \\ p^{\epsilon}_{\cdot o} &= c^1 p^{\epsilon}_{\cdot 1} = c^2 p^{\epsilon}_{\cdot 2} = c^3 p^{\epsilon}_{\cdot 3} \end{aligned} \tag{97}$$

where k_i and c^i are constants the proper figure[23] degenerates into a proper figure of points and planes.[16] In the special case in which

$$p^k_{.\lambda} \equiv 0, \qquad p^o_{.o} \neq 0 \tag{98}$$

the proper figure[23] degenerates into a semi-proper figure of points and planes.[17]

The general sum of semi-improper figures of points and planes[18] is a *semi-improper figure.*[24] This semi-improper figure[2] can be decomposed into three semi-improper figures and one improper figure of points and planes[21] with respect to any three non-parallel proper planes. It can also be decomposed into three semi-improper figures of points and planes[18] with respect to any three distinct improper points. If

$$\begin{aligned} p^1_{.\lambda} &= k_2 p^2_{.\lambda} = k_3 p^3_{.\lambda} \\ p^k_{.o} &= c^1 p^k_{.1} = c^2 p^k_{.2} = c^3 p^k_{.3}\,. \end{aligned} \tag{99}$$

This semi-improper figure[24] degenerates into a semi-improper figure of points and planes.[18] If

$$p^k_{.l} \equiv 0 \tag{100}$$

the semi-improper figure[24] degenerates into an improper figure of points and planes.[21]

The general sum of semi-attached figures of points and planes,[19] is a *semi-attached figure.*[25] It can be decomposed into a sum of three semi-attached figures of points and planes[19] with respect to any three non-parallel free planes. If

$$\begin{aligned} p^o_{.i} &= k_1 p^1_{.i} = k_2 p^2_{.i} = k_3 p^3_{.i} \\ p^k_{.1} &= c^2 p^k_{.2} = c^3 p^k_{.3} \end{aligned} \tag{101}$$

where k_i and c^i are constants, the semi-attached figure[25] degenerates into a semi-attached figure of points and planes.[19]

The general sums of free figures of points and planes[20] is a free figure.[26] It can be decomposed into three free figures of points and planes[20] with respect to any three non-parallel free vectors or any three non-parallel free planes. If

$$\begin{aligned} p^1_{.l} &= k_2 p^2_{.l} = k_3 p^3_{.l} \\ p^k_{.1} &= c^2 p^k_{.2} = c^3 p^k_{.3} \end{aligned} \tag{102}$$

the free figure[26] degenerates into a free figure of points and planes.[20]

Sums of proper, semi-improper, semi-attached, or free figures of this type[22] are again proper, semi-improper, semi-attached, or free figures of the same type.[22] They are added when their decomposition with respect to the same proper points, or free vectors, proper planes, or free planes are added. When such a figure of points and planes[22] is multiplied by a scalar the affine normalizations of the figure of points and planes[15] into which it may be decomposed are

multiplied by a scalar. Addition of these figures[22] satisfies the commutative, associative, and distributive laws.

The Affine Ancestry of the Figures of Study

Among the figures which have been discussed are the affine ancestors of all of the Study figures. Without precisely defining the term affine ancestor, this section will show the exact relation of these two sets of figures.

The first figure which is described in the "Geometrie der Dynamen" is the "Linienkreuz." Each of the Study figures is supported by a proper or improper "Linienkreuz." The proper "Linenkreuz" is the figure of a line in space and the family of planes orthogonal to it, while the improper "Linenkreuz" consists of a direction in space and the family of planes orthogonal to it. This figure has neither polarization nor normalization. It has many affine ancestors. All of the figures representing the supports of affine figures are its ancestors. Of course, there is no orthogonality property in affine space. Thus there are the line support of the proper figure of points[2] and of the proper figure of planes[9], the free plane support of the improper figure of points[4], and semi-free figure of planes[11] the free line support of the free figure of planes, as well as the combinations of proper and improper points and planes which are supports of the figures of points and planes.[16]

The "Stab" is the figure of two proper points and is supported by a proper "Linienkreuz." An inner polarization is defined by the order in which the points are taken, and the normalization is the distance between the two points. The proper figure of points[2] is the affine ancestor of the "Stab." It is the figure of two points and has the same type of inner polarization. Its normalization is the separation of two points which is an affine concept distinct from that of distance in metric space.

The "Stäbepaare" is the figure of two improper points or free vectors and is supported by an improper "Linienkreuz." An inner polarization is defined in a plane by the order in which the two vectors are taken and the area of the plane segment enclosed by the two vectors gives the inner normalization. The affine ancestor is the improper figure of points[4] which is identical in all respects, except that its normalization is an affine area instead of a metrical one.

The "Keil" is the figure of two planes and is supported by a proper "Linienkreuz." An outer polarization is defined about the line of support by the order in which the planes are taken, and the normalization is the tangent of the angle of separation of the planes. The affine ancestor of the proper "Keil" is the proper figure of planes.[9] In support and polarization it is identical with its descendent. Its normalization is the opening which is an affine concept which is closely allied with the tangent in metric space.

The improper "Keil" is the figure of two parallel planes and has as support an improper "Linienkreuz." Its polarization is the order in which the planes are taken while the normalization is the distance between the planes. Its affine ancestor is the semi-free figure of planes,[11] identical in support and polarization, but whose normalization is the affine property, separation, instead of the metric property of distance.

Another figure is defined by the sum of "Stäbe" or the sum of "Keile". This is the "Motor" which is defined by Study in terms of two lines which are both proper and may be orthogonal. It has associated with it two normalizations and a screw sense given by the order in which the lines are taken. Its length is the distance between the two lines and its second normalization is the tangent of the angle between the two lines. The common normal of these lines may be thought of as the support of the figure. In the special cases in which the lines are parallel or intersect, the motor degenerates into two other figures, which are respectively the "Translator" and the "Rotor." The affine ancestors of the "Motor" are the proper sums of figures of points[6] and the proper sums of figures of planes.[14] These figures are identical except in their normalization. The normalizations of these figures are a separation or opening and an affine area or a separation of planes. In the special cases in which the tensors representing them satisfy Plücker's relation they degenerate into the "Rotor" of Study. In the second special case in which the components $p^{oi} \equiv 0$ or the components $p_{ij} \equiv 0$, they degenerate into the "Translator" of Study. The condition that the lines in terms of which Study defines his "Motor" may not be orthogonal appears only in metric space. However, the condition introduced by Study's restrictions on the lines is essentially that the normalizations may not become infinite, and that is already inherent in the affine figures.

The figure of a proper point and a plane is called the "Quirl." Its support is the proper "Linienkreuz." The polarization is defined by the order of point and plane while the normalization is the reciprocal of the distance between the point and plane. The affine ancestor of the proper "Quirl" is the proper figure of a point and a plane.[16] It is the figure of a proper point and of a proper plane in affine space. Since it is not possible to associate a normal line with a plane in affine space, this figure does not yet have a line support. Only in metric space does it gain this freedom. The polarizations are identical and the affine normalization is closely related to the concept of distance between a point and a plane in metric space.

The improper "Quirl" is the figure of an improper point or free vector and a plane. Its support is an improper "Linienkreuz," and its polarization is given by the order in which free vector and plane are taken. The normalization is the cotangent of the angle between the free vector and the plane. The affine ancestor of this figure is the semi-improper figure of points and planes.[18] Both figures are identical in polarization and differ in support and normalization in a manner similar to that described in the previous paragraph.

The sum of "Quirle" is a new figure, called by Study the "Impulsor." Like the "Motor," it is defined in terms of two lines and has the same type of support and polarization. Its two normalizations are the same; its length is the distance between the two lines, and its second normalization is the tangent of the angle between the lines. In the special case in which the two lines intersect at right angles, the figure is called a "Torsor." If one of the lines becomes improper, the figure is called an "Ejector." The proper figure of sums of figures of points and planes[23] is the affine ancestor of the "Impulsor." Since it does not have a line support it is not too similar to the metric descendant. If a decomposition

with respect to a given proper point is considered the polarization and normalization may be discussed. Note that they are much more complex ones than are ordinarily considered. In the special case in which the figure[23] is of rank one it degenerates to the ancestor of the "Torsor" and in the special case in which it is a tensor of rank one whose only non-zero components are p^i_λ it becomes the affine ancestor of the "Ejector."

This completes the tensor interpretation of the affine ancestors of the Study figures. In a later paper the tensor interpretation of the Study figures themselves in metric space will be considered.

Appendix

Names of the Affine Figures

As can be seen from the preceding pages, it is difficult if not impossible to give a clear discussion of the geometric figures of affine space without naming them. The procedure has been to write down the description of the figure, such as "the figure of two proper points", which is cumbersome and lacking in precision.

There is considerable disagreement on the naming of the figures. One method, which has been suggested by others, is to use the names of Study with such additional words as are necessrry. This gives names such as "Motor", "Rotor," and "Tractor". Four disadvantages result:

(a) Confusion with the same words used in technology.

(b) Confusion with the same words used with quite different meanings by such mathematicians as Ball, von Mises, Klein, and MacAuley.

(c) Confusion with the figures discussed by Study, which are distinct from the affine figures.

(d) The proposed schemes do not have names for all of the figures treated in this paper.

The set of names used in the development of this paper overcomes all of the foregoing difficulties. The names are based on Greek words closely associated with the names of Study. The words are easy to use and they have the great advantage that they do not convey any preconceived ideas. The proposed names are as follows:

1. A figure whose tensor representation is an alternating contravariant tensor of valence two that satisfies Plücker's relation is called a *rhabdel.*

2. The *proper rhabdel* is a figure determined by a pair of distinct proper points.

3. A *semi-proper rhabdel* is the figure determined by a proper point and a free vector.

4. The *improper rhabdel* is the figure determined by two free affine vectors.

5. A figure whose tensor representation is an alternating contravariant tensor of valence two which does not in general satisfy Plücker's relation is called a *kineal.*

6. The generalized sum of proper and improper rhabdels is a *proper kineal.*

7. The generalized sum of semi-proper and improper rhabdels is a *semi-proper kineal.*

8. A figure whose tensor representation is an alternating covariant tensor of valence two that satisfies Plücker's relation is called a *strophel.*

9. A *proper strophel* is a figure determined by a pair of distinct non-parallel proper planes.

10. A *free strophel* is a figure determined by a pair of distinct free planes.

11. A *semi-free strophel* is a figure determined by one improper plane and one free plane.

12. A *semi-attached strophel* is a figure determined by a proper plane and a free plane that are not parallel or coincident.

13. A figure whose tensor representation is an alternating covariant tensor of valence two that does not in general satisfy Plücker's relation is called a *helissel.*

14. The generalized sum of proper, free, and semi-free strophels is a proper *helissel.*

15. A figure whose tensor representation is a mixed tensor of valence two and rank one is called a *charybdel.*

16. A *proper charybdel* is a figure determined by a proper point and a proper plane which are not coincident.

17. A *semi-proper charybdel* is the figure determined by a proper point and an improper plane.

18. A *semi-improper charybdel* is the figure determined by an improper point and a proper plane.

19. A *semi-attached charybdel* is the figure determined by a proper point and a free plane.

20. A *free charybdel* is the figure determined by a free vector and a free plane.

21. An *improper charybdel* is the figure determined by an improper point and an improper plane.

22. A figure whose tensor representation is a mixed tensor of valence two and rank four is called a *dynatel.*

23. The generalized sum of proper charybdels is a *proper dynatel.*

24. The generalized sum of semi-improper charybdels is a *semi-improper dynatel.*

25. The generalized sum of semi-attached charybdels is a *semi-attached dynatel.*

26. The generalized sum of free charybdels is a *free dynatel.*

MEASURABLE FUNCTIONS

By Philip Franklin[1]

The first part of this paper is expository. We begin with elementary proofs of a series of theorems on the approximation of measurable functions, summable functions, and Riemann integrable functions. Although many of these theorems are well known, they are not sufficiently emphasized in most treatments. Collectively, they throw light on the nature of summable functions.

In connection with these, we discuss Carathéodory's classes of bounded measurable functions on a finite interval.[2] His classification depends on the region of definition used. We give a modification which is free from this objection, and consider its application to functions on infinite intervals.

We also define, uniquely for a set of equivalent functions, two functions either of which may represent the set. One is an *lu* function,[3] the upper limit of an increasing sequence of lower semi-continuous functions, and the other is a *ul* function. Their relation to Kempisty's approximate limits[4] is discussed. We show that they correctly represent the function as to Baire classification,[5] being in this sense of the smoothest possible type.

We indicate how our results may be extended to functions of more than one variable.

Finally, we prove two properties of functions equivalent in the sense of Besicovitch,[6] as used for his B.a.p. functions. We show that a continuous B.a.p. function may be found equivalent to any B.a.p. function, and that if a sequence of B.a.p. functions is B-convergent, it is B-convergent to some B.a.p. function.

1. **Upper approximation to bounded functions.** We begin by showing that, *for any bounded and measurable function $f(x)$, we may find a lower semi-continuous function $\varphi(x)$ such that*

$$\varphi(x) > f(x)$$

$$\int_a^b [\varphi(x) - f(x)]\,dx < \epsilon$$

To construct $\varphi(x)$, we consider intervals of length δ on the ordinates, and the corresponding sets of values of x,

(1) $\qquad e_n \quad \text{where} \quad n\delta \leqq f(x) < (n+1)\delta$

[1] Presented to the American Mathematical Society.

[2] C. Carathéodory, Vorlesungen über Reelle Funktionen, chapter VII.

[3] For properties of upper and lower semi-continuous functions, see Carathéodory, l. c.[2], chapter III or E. W. Hobson, Theory of functions of a Real Variable and the Theory of Fourier's Series, vol. I, chapter V, vol. II, chapter II.

[4] S. Kempisty, Fundamenta Mat. vol. VI (1924), p. 6.

[5] For a discussion of Baire's classification see[2], Hobson, l. c.[2], vol. II, chapter IV or Ch. J. de la Vallée-Poussin, Intégrales de Lebesgue, fonctions d'ensemble, classes de Baire (Borel monograph).

[6] A. S. Besicovitch, Almost Periodic Functions (Monograph, Cambridge University Press).

as well as the sets

$$E_n \quad \text{where} \quad n\delta \leqq f(x) \tag{2}$$

If N is a bound for $f(x)$,

$$|f(x)| < N, \tag{3}$$

and

$$n_1\delta \geqq N,$$

we have

$$\sum_{n}^{n_1} e_i = E_n .$$

Now embed each set E_n in an open set O_n, such that

$$M(E_n) < M(O_n) < M(E_n) + \eta_n . \tag{4}$$

Since each set E_n includes the set E_{n+1}, we may omit that part of O_{n+1} which is outside of O_n, and so assume that O_n includes O_{n+1}. Replace O_n by $O_{-n_1} \cdot O_{-n_1+1} \cdots O_{n-1} \cdot O_n$, again an open set.

Each set O_n consists of an enumerable number of non-overlapping open intervals. Let $L_n(x) = 1$ on these intervals and zero elsewhere.

If one of these intervals is $a_i < x < b_i$, we may define a function by

$$\begin{aligned} a_i \leqq x \leqq a_i + h_i ; \quad & T_i(x) = \frac{x - a_i}{h_i} \\ a_i + h_i \leqq x \leqq b_i - h_i ; \quad & T_i(x) = 1 \\ b_i - h_i \leqq x \leqq b_i ; \quad & T_i(x) = \frac{b_i - x}{h_i} . \end{aligned} \tag{5}$$

The function c_k equal to $T_i(x)$ on the first k intervals of O_n and zero elsewhere is continuous. If we put $h_i = (b_i - a_i)/k$ for c_k, and let k become infinite, the upper limit of $c_k(x)$ will be $L_n(x)$. Thus L_n is lower semi-continuous. Finally we put

$$\phi(x) = \delta \sum_{-n_1}^{n_1} L_n(x) - n_1\delta \tag{6}$$

We shall now show that this has the desired properties, if δ and the η_n are properly chosen.

a) The function $\phi(x)$ is lower semi-continuous, being the sum of a finite number of lower semi-continuous functions L_n.

b) If x is in e_m, it is in E_n for all $n \leqq m$, which are $n_1 + m + 1$ in number. Thus x is in an interval of O_n, and hence $L_n(x) = 1$ for each of these values. For other n, $L_n(x) \geqq 0$, so that

$$\phi(x) \geqq \delta(n_1 + m + 1) - n_1\delta \geqq \delta(m + 1) > f(x). \tag{7}$$

c) By definition of the Lebesgue integral,

$$\sum n\delta M(e_n) \leqq \int_a^b f(x)\,dx < \sum (n+1)\delta M(e_n) \tag{8}$$

But, since the e_n are non-overlapping

$$\sum_{i=n}^{n_1} M(e_i) = M(E_n)$$

and

$$\sum_{-n_1}^{n_1} M(E_n) = \sum (n_1 + n + 1)M(e_n) = \sum nM(e_n) + (n_1+1)(b-a). \tag{9}$$

Now recall the definition of $L_n(x)$ and $\phi(x)$.

$$\int_a^b L_n(x)\,dx = 1[M(O_n)] \leqq M(E_n) + \eta_n,$$

by (4). Hence from (6)

$$\int_a^b \phi(x)\,dx \leqq \delta \sum_{-n_1}^{n_1} M(E_n) + \delta \sum \eta_n - n_1\delta(b-a).$$

But from (8) and (9),

$$\int_a^b f(x)\,dx \geqq \delta \sum_{-n_1}^{n_1} M(E_n) - (n_1+1)\delta(b-a).$$

This shows that

$$\int_a^b \phi(x)\,dx - \int_a^b f(x)\,dx \leqq \delta \sum \eta_n + \delta(b-a).$$

The right number may be made less than ϵ by a suitable choice of δ and the η_n, e.g.

$$\eta_n < \epsilon/2^{|n|+3}, \qquad \delta < \epsilon/4(b-a), \qquad \delta < 1. \tag{10}$$

Thus, since for all x, $\phi(x) \geqslant f(x)$ by (7), we have finally

$$0 \leqq \int_a^b |\phi(x) - f(x)|\,dx = \int_a^b [\phi(x) - f(x)]\,dx < \epsilon. \tag{11}$$

The inequality holds a fortiori for any upper limit x in the interval $a < x < b$.

2. **Lower approximation to bounded functions.** We may define a lower approximation to $f(x)$ as an upper semi-continuous function $\psi(x)$, such that

$$\psi(x) < f(x)$$

$$\int_a^b [f(x) - \psi(x)]\,dx < \epsilon.$$

We obtain this by applying the previous construction for $\phi(x)$ to $-f(x)$, giving a lower semi-continuous function $-\psi(x)$ such that

$$-\psi(x) > -f(x)$$

and

$$\int_a^b [-\psi(x) + f(x)]\,dx < \epsilon.$$

the desired conditions.

3. **Degree of approximation.** To estimate the degree of approximation, we notice that with each set O_n we may associate the set $O_n - E_n$. Each of these sets is of measure less than η_n. Again, by (10), the sum of these sets for all n, S, is of measure less than ϵ. Let us consider any point of the fundamental interval not in S. It is in some set e_m, and hence in all E_n with subscripts $n \leqq m$. Consequently for these $n_1 + m + 1$ values it will be in the corresponding O_n, and hence for this x and all such n, $L_n(x) = 1$. Again, the point will not be in any of the sets $O_n - E_n$, if $n > m$, and therefore not in any other O_n, with $n > m$. Thus for values of $n > m$, $L_n(x) = 0$. From the definition of $\phi(x)$ given in (6) we have in this case

$$\phi(x) = \delta[n_1 + m + 1] - n_1\delta = (m + 1)\delta.$$

But, as x is in e_m,

$$m\delta \leqq f(x) < (m + 1)\delta,$$

so that

$$0 < \phi(x) - f(x) \leqq \delta \tag{12}$$

This proves that, *except in S, $\phi(x)$ approximates $f(x)$ to within δ.* Since δ is only subject to (10), it may be arbitrarily small.

Similar remarks apply to $\psi(x)$.

We may summarize the results so far obtained in:

THEOREM I. *For any bounded measurable function $f(x)$ ($|f(x)| < N$), defined for $a \leqq x \leqq b$, and a preassigned positive quantity ϵ, there exists two semi-continuous functions $\phi(x)$ and $\psi(x)$, each numerically at most N, such that*

$$\psi(x) < f(x) < \phi(x) \tag{13}$$

for all x, and except for a set of values of x, S, of measure less than ϵ, each of these functions approximates $f(x)$ uniformly to within ϵ; that is

$$f(x) - \psi(x) < \epsilon, \qquad \phi(x) - f(x) < \epsilon \tag{14}$$

except on S, where $M(S) < \epsilon$.

Finally, for any x between a and b, inclusive,

$$\int_a^x [\phi(x) - f(x)]\,dx < \epsilon \tag{15}$$

and

$$\int_a^x [f(x) - \psi(x)]\,dx < \epsilon \tag{16}$$

4. **Sequences of approximate functions.** Let us start with a bounded function $f(x)$, and a sequence of quantities $\epsilon_1, \epsilon_2, \epsilon_3, \cdots$ approaching zero. Let $\psi_n(x)$ and $\phi_n(x)$ be the semi-continuous functions and S^n the exceptional set corresponding to ϵ_n whose existence is given by Theorem I. However, in constructing these, let us use a succession of S^n's such that δ_n is an integral multiple of δ_{n+1}. Then, at any stage if $\delta_n = k\delta_{n+1}$, the set e_m^n will be replaced by k sets e_m^{n+1} where $m = km, k(m+1), \cdots, k(m+k-1)$, such that

$$e_m^n = \sum e_m^{n+1}$$

Each set E_m^{n+1} will be contained in E_m^n. We may arrange that O_m^{n+1} is contained in O_m^n, and moreover that the enlargements of intervals of O_m^{n+1} are contained in the enlarged intervals of O_m^n. Under these conditions, we will have

$$\phi_n(x) > \phi_{n+1}(x), \qquad \psi_n(x) < \psi_{n+1}(x) \tag{17}$$

for all x, and incidentally

$$S^{n+1} < S^n. \tag{18}$$

Let $\bar{S}$ be the set of points common to all the S^n

$$\bar{S} = S^1 \cdot S^2 \cdot S^3 \cdots$$

Then

$$M(\bar{S}) = \lim M(S^n) = \lim \epsilon_n = 0. \tag{19}$$

Consider any point x not in $\bar{S}$. It must be outside of some S^n, say S^N, and hence outside of all subsequent S^n. Thus, for $n > N$, we have

$$[\phi_n(x) - f(x)] < \epsilon_n$$

so that

$$\lim \phi_n(x) = f(x) \qquad \text{except on } \bar{S}. \tag{20}$$

Similarly

$$\lim \psi_n(x) = f(x) \qquad \text{except on } \bar{S}. \tag{21}$$

Since the values of $\phi_n(x)$ steadily decrease with n, and always exceed $-N$, they approach a limit at all points. Let us put

$$\lim \phi_n(x) = \bar{\phi}(x) \tag{22}$$

and similarly for the increasing values of $\psi_n(x)$,

$$\lim \psi_n(x) = \bar{\psi}(x). \tag{23}$$

Then we have two functions $\bar{\phi}(x)$ and $\bar{\psi}(x)$ with $\bar{\psi}(x) \leqq f(x) \leqq \bar{\phi}(x)$ for all x, each of Baire's second class and each of which can differ from $f(x)$ only on the set $\bar{S}$, of measure zero. We say that these functions are *equivalent* to $f(x)$.

We also note that outside of any set S^n, which may be taken of arbitrarily small measure, the functions $\phi_n(x)$ and $\psi_n(x)$ approach $f(x)$ uniformly. This is in accord with Egoroff's theorem.

We collect these facts in

THEOREM II. *Every bounded measurable function $f(x)$ is equivalent to two functions of Baire's second class, and it is intermediate to them for all values of x.*

5. **Infinite intervals and unbounded functions.** If $f(x)$ is *positive*, $f(x) \geqq 0$, unbounded and measurable on an infinite or semi-infinite interval, we may obtain an approximation theorem similar to Theorem II as follows. We define truncated functions $F_n(x)$ as follows:

$$
\begin{aligned}
&F_n(x) = 0, && x \leqq -n \quad \text{or} \quad x \geqq n \\
&F_n(x) = f(x), && -n < x < n \quad \text{and} \quad 0 \leqq f(x) \leqq n \\
&F_n(x) = n, && -n < x < n \quad \text{and} \quad f(x) > n
\end{aligned} \tag{24}
$$

The function $F_n(x)$ is bounded, and on that part of the finite interval $-n < x < n$ outside of S^n may be approximated from below to within ϵ_n by a semi-continuous function $\psi_n(x)$. We make this zero for $x \leqq -n$ and $x \geqq n$. If the construction is similar to that previously used, the functions $\psi_n(x)$ will increase with n, and so approach a limiting function

$$\lim \psi_n(x) = \bar{\psi}(x)$$

which may, of course, be infinite. The exceptional sets S^n will have added parts between n and $n + 1$, and $-n$ and $-n - 1$, but we may arrange that that part of S^{n+1} in the interval $-n < x < n$ is included in S^n. Then any x not in all the sets S^n, i.e., not in $\bar{S}$, a set of measure zero, will for $n > N > |x|$, where x is outside of S^N, be outside of all the S^n. Thus, for this x, we will have

$$\bar{\psi}(x) = f(x).$$

As this holds except on a set of measure zero, $f(x)$ is equivalent to $\bar{\psi}(x)$.

For a function of unrestricted sign, we write

$$f(x) = p(x) + n(x), \tag{25}$$

where

$$
\begin{aligned}
&p(x) = f(x), && \text{if} \quad f(x) > 0 \\
&p(x) = 0, && \text{if} \quad f(x) \leqq 0
\end{aligned}
$$

and

$$
\begin{aligned}
&n(x) = f(x), && \text{if} \quad f(x) < 0 \\
&n(x) = 0, && \text{if} \quad f(x) \geqq 0.
\end{aligned}
$$

Then if $f(x)$ is measurable, $p(x)$ and $n(x)$ will each be measurable. By the argument for a positive function, we may find a function $P(x)$ of Baire's class two, the $\bar{\psi}(x)$ above, equivalent to $p(x)$. Similarly there is an $N(x)$, the $-\bar{\psi}(x)$ for $-n(x)$, equivalent to $n(x)$. Thus, finally,

$$B(x) = P(x) + N(x) \tag{26}$$

is a function of Baire's class two, equivalent to $f(x)$. This proves the well known fact:

THEOREM III. *Every measurable function (on an infinite or finite interval) is equivalent to a function of Baire's second class.*

COROLLARY. *If the function $f(x)$ is summable on the interval of integration, which may be infinite, we may write:*

$$\int_a^b f(x)\,dx = \int_a^b B(x)\,dx.$$

6. **Approximations by continuous functions.** In the previous discussions we were careful to have our approximating functions steadily increase, or decrease so that a limit was approached at all points, including those of the exceptional set. If we give up this last requirement, we do not obtain an equivalent function, but may dispense with one limiting process. Let us recall the construction of section 1, and instead of using the lower-semi-continuous function $L_n(x)$, use the corresponding continuous functions $C_n(x)$, which result when only k of the enlarged intervals are taken. By taking k sufficiently large, the sum of the omitted intervals will not exceed η_n. This leads us to define the continuous function

$$\alpha(x) = \delta \sum_{-n_1}^{n_1} C_n(x) - n_1\delta \tag{27}$$

We regard the exceptional set S here as made up of the sets $O_n - E_n$, and the enlargements previously used, as well as of the omitted intervals. The reasoning of section 3 still applies to values of x not in the exceptional set, and show that

$$0 < \alpha(x) - f(x) < \delta$$

for points not in S, but for points in S we have merely

$$\alpha(x) < N.$$

We may form a function $\beta(x)$, similar to $\alpha(x)$, approximating $f(x)$ from below. We summarize the facts here in

THEOREM IV. *For any bounded measurable function $f(x)$ ($|f(x)| < N$), defined for $a \leqq x \leqq b$, and a given positive quantity ϵ, there exist two continuous functions $\alpha(x)$ and $\beta(x)$, each numerically at most N, such that*

$$\beta(x) < f(x) < \alpha(x) \tag{28}$$

and

$$f(x) - \beta(x) < \epsilon, \qquad \alpha(x) - f(x) < \epsilon$$

except on S, where

$$M(S) < \epsilon.$$

Finally, for any x between a and b, inclusive

$$\int_a^x |\alpha(x) - f(x)|\, dx < \epsilon$$

and

$$\int_a^x |f(x) - \beta(x)|\, dx < \epsilon.$$

Let us next recall the construction of section 5, for unbounded functions on infinite intervals. For a positive function of this type, we define $F_n(x)$ by (24), and form a function $\beta_n(x)$ for it. Since we may regard the exceptional sets S^n as similar to those previously used, together with omitted intervals, and at each stage retain all intervals not previously omitted, we may arrange that S^n includes that portion of S^{n+1} in the interval $-n < x < n$, as before. This will lead to a set of functions $\beta_n(x)$, approaching $f(x)$ outside of $\bar{S}$. For a function of unrestricted sign, we put as before

$$f(x) = p(x) + n(x)$$

and form a sequence of functions $\beta_n(x)$ for $p(x)$, and $\alpha_n(x)$ for $n(x)$, where $-\alpha_n(x)$ is analogous to $\beta_n(x)$ for $-n(x)$. We may then put

$$\gamma_n(x) = \alpha_n(x) + \beta_n(x),$$

and so obtain a sequence of continuous functions approximating $f(x)$ where $f(x)$ is finite and becoming infinite where $f(x)$ is ∞. That is,

THEOREM V. *Every measurable function $f(x)$ (on an infinite or finite interval) is, except on a set $\bar{S}$ of measure zero, the limit of a sequence of continuous functions, $\gamma_n(x)$.*

If the positive function $f(x)$ is summable on the infinite integral, for n sufficiently large, say $> n_1$ we will have

$$\int_{-\infty}^{\infty} |f(x) - F_n(x)| < \epsilon, \qquad n > n_1.$$

If we choose the ϵ_n properly, we may arrange that

$$\lim n\epsilon_n = 0.$$

We now select $n > n_1$, and also so large, say $> n_2$, that

$$4n\epsilon_n < \epsilon, \qquad n > n_2.$$

Then

$$\int_{-\infty}^{\infty} |F_n(x) - \beta_n(x)|\, dx = \int_{-n}^{n} |F_n(x) - \beta_n(x)|\, dx \leqq 4n\epsilon_n < \epsilon,$$

since the integrand is less than $n + 2\delta < 2n$ on the exceptional set of measure less than ϵ_n, and on the remaining set of measure less than $2n$, the integrand is less than ϵ_n. Thus

$$\int_{-\infty}^{\infty} |f(x) - \beta_n(x)|\, dx < 2\epsilon.$$

For any function this applies to

$$|p(x) - \beta_n(x)| \quad \text{and} \quad |n(x) - \alpha_n(x)|,$$

and proves the

COROLLARY. *If the function $f(x)$ is summable on the interval of integration, which may be infinite, we have*

$$\int_a^b f(x)\, dx = \lim \int_a^b \gamma_n(x)\, dx$$

and

$$\int_a^b |f(x) - \gamma_n(x)|\, dx < \epsilon, \qquad n > n_\epsilon.$$

7. **Reimann-Integrable functions.** The usual definition of Riemann sums for a bounded function $f(x)(|f(x)| \leqq N)$, on a finite interval is:

$$\sum M_i \Delta_i \quad \text{and} \quad \sum m_i \Delta_i$$

where the Δ_i are intervals, k in number, whose maximum length approaches zero, and the M_i, m_i are greatest and least upper bounds in the intervals Δ_j. If we take intervals of width $2h$ about each end point of an interval Δ_i, we may define a continuous function $\phi(x)$ as follows. The function $\phi(x) = M_i$ in the intervals Δ_i, diminished by the intervals of width h. At a point on the side of the greater M_i, we put $\phi(x) = M_i$ in the h interval. Otherwise we make $\phi(x)$ a linear function. Similarly a continuous function $\psi(x)$ may be set up from the values of m_i.

Evidently

$$\psi(x) \leqq f(x) \leqq \phi(x), \tag{29}$$

and

$$\int |\phi(x) - \psi(x)|\, dx \leqq \sum (m_i - m_i)\Delta_i + 2khN \tag{30}$$

When the function is Riemann integrable, the first sum here approaches zero with maximum Δ_i. A necessary and sufficient condition for this, due to Lebesgue, is that the function $f(x)$ be continuous except at a set of points of measure zero.

The second sum approaches zero with kh.

Let us form a sequence of functions $\phi_n(x)$ and $\psi_n(x)$, where in each case the new intervals are obtained by subdividing the old, and choose decreasing h_n so that $k_n h_n$ approaches zero. Then the values of $\phi_n(x)$ will steadily decrease,

and so tend to a limiting function $\bar{\phi}(x)$, while $\psi_n(x)$ will steadily increase to a limiting function $\bar{\psi}(x)$. From (29) and (30).

$$\bar{\psi}(x) \leqq f(x) \leqq \bar{\phi}(x),$$

and

$$\int | \bar{\phi}(x) - \bar{\psi}(x) | \, dx = 0.$$

Consequently, except on a set of measure zero, we will have

$$\bar{\psi}(x) = f(x) = \bar{\phi}(x),$$

so that we have proved:

THEOREM VI. *Every bounded Riemann-integrable function is equivalent to an upper semi-continuous function which always exceeds or equals it, and a lower semi-continuous function which it always exceeds or equals.*

We note that the construction just used is essentially that of section 1, with the open sets O_n replaced by finite sets of intervals. This change is made possible by the Riemann integrability of our function.

If we had a positive unbounded function on an infinite interval, continuous except on a set of measure zero, and we applied the process of section 5 to it, each function $F_n(x)$ would be Riemann-integrable. Hence we could by the argument for Theorem VI, find a continuous function $\psi_n(x)$ approximating $F_n(x)$ from below to within ϵ_n except in a set of measure less than ϵ_n. This leads to a sequence of continuous functions $\psi_n(x)$, such that

$$\lim \psi_n(x) = \bar{\psi}(x),$$

since the $\psi_n(x)$ increase, and $\bar{\psi}(x)$ is equivalent to $f(x)$. The argument of section 5 then applies unchanged to give:

THEOREM VII. *Every (unbounded) function on an infinite interval, continuous except on a set of measure zero, is equivalent to a function of Baire's first class. In particular, if every truncated function obtained from $f(x)$ or $| f(x) |$ is Riemann integrable, then $f(x)$ is equivalent to a function of Baire's first class.*

The equivalent function may be regarded as the sum of a lower semi-continuous and an upper semi-continuous function.

8. **Classification of Functions.** Carathéodory has given a classification of finite functions measurable on finite intervals into four classes, as follows:

I. Functions which, on their region of definition are equivalent to continuous functions.

II. Functions which are continuous on a subregion of their region of definition of the same measure.

III. Functions which are equivalent to a function of Baire's class one on their region of definition.

IV. Unrestricted measurable functions.

He shows that each of these classes is more restricted than that which follows,

i.e. each class contains all functions of the preceding classes, and contains some functions not in the earlier class.

This classification depends essentially on the region of definition. In fact, in some cases, omitting a set of zero measure may change the class. Thus by Theorem V, if the set S be omitted, any measurable function becomes of class III. Again, directly from the definition, every function of class II becomes of class I, if an appropriate set of measure zero be omitted.

We shall set up a classification for functions defined on sets of positive measure which avoids this difficulty. We always take the region used in the classification as the smallest finite or infinite interval including all points for which the function is defined, and consider the function as zero for all points in this interval at which the function was originally undefined. Thus our classes are:

I. Functions equivalent on some interval to a continuous function.

II. Functions, not in class I, equivalent on some interval to a bounded function whose discontinuities form a set of measure zero.

III. Functions, not in classes I or II, equivalent on some interval to a function of Baire's first class.

IV. All measurable functions not of the first three classes.

We observe that, by Lebesgue's theorem, the functions of class II, if on finite intervals, are equivalent to Riemann integrable functions.

9. **Semi-continuity and the first two classes.** A function of our class I equivalent to a continuous function on an interval may be uniquely represented by this continuous function, since two continuous functions are equal if equivalent. As lower semi-continuous functions, or l-functions and upper semi-continuous functions, or u-functions are both classes including continuous functions, and conversely a function which is both l and u is continuous, in terms of semi-continuity *the functions of our class I are those equivalent to a function which, itself, is both l and u.*

A function equivalent to two functions, f_1 an l-function and f_2 a u-function on an interval is of our class II. For, let $\psi_n(x)$ be an increasing sequence of continuous functions approaching f_1, and $\phi_n(x)$ be a decreasing sequence of continuous functions approaching f_2. Then, except on a set of measure zero, $\bar{x}$,

$$\psi_n(x) \leqq f_1(x) = f_2(x) \leqq \phi_n(x), \qquad x \neq \bar{x}.$$

Hence, since $\psi_n(x)$ and $\phi_n(x)$ are continuous,

$$\psi_n(x) \leqq \phi_n(x)$$

at all points. For, if $\psi_n(x) - \phi_n(x)$ were positive at any point, it would be positive on a set of positive measure. From the last inequality, we have

$$f_1(\bar{x}) \leqq f_2(\bar{x}).$$

If $f_3(x)$ is any function such that $f_1(x) \leqq f_3(x) \leqq f_2(x)$, then

$$f_3(x + h) \geqq f_1(x + h) \geqq f_1(x) - \epsilon \quad \text{since} \quad f_1(x) \text{ is } l.$$

Similarly $f_3(x+h) \leqq f_2(x+h) \leqq f_2(x) + \epsilon$, since $f_2(x)$ is u. Hence $f_3(x)$ is continuous at all points where $f_1(x) = f_2(x)$ or except at a set of measure zero. Thus it is of our class II.

10. **Equivalent u-functions.** Let $f(x)$ be a bounded measurable function. If $F(x) = \int_a^x f(x)\,dx$, the derived numbers for F, D^+F (or D^-F, D_+F, D_-F) yield measurable functions, bounded above. The upper bound of any one of these in any open interval is the same as that of

$$\frac{1}{x_2 - x_1}\int_{x_1}^{x_2} f(x)\,dx = \frac{F(x_2) - F(x_1)}{x_2 - x_1}$$

for x_1 and x_2 in the open interval. As an upper limit function, it is u. The lower bound of these, for all open intervals including a given point x, defines a function $f_u(x)$, which is a u-function, since it is the lower bound of a set of u-functions.

This function $f_u(x)$ has the following properties:

a) $f_u(x)$ is the same for two equivalent functions.

b) If f_1 and f_2 are equivalent to two functions $\bar{f}_1$ and $\bar{f}_2$ such that $\bar{f}_1 \geqq \bar{f}_2$ at all points, then the corresponding f_u's satisfy the same relation.

c) At a point of continuity, x_0, $f_u(x_0) = f(x_0)$.

This follows from $f(x+h) = f(x_0) + \theta\epsilon$ which leads to

$$\frac{1}{x_2 - x_1}\int_{x_1}^{x_2} f(x)\,dx = f(x_0) + \theta_1\epsilon.$$

Let $f(x)$ be equivalent to $\bar{f}(x)$, a u-function which is the lower limit of a sequence of continuous functions $\psi_n(x)$. And let $\psi(x)$ be any continuous function which exceeds $f(x)$. Then from c) $\psi_u(x) = \psi(x)$ and $[\psi_n(x)]_u = \psi_n(x)$. Also from a) $f_u(x) = \bar{f}_u(x)$, and from these facts and b) we have

$$f_u(x) = \bar{f}_u(x) \leqq [\psi_n(x)]_u = \psi_n(x) \quad \text{and} \quad f_u(x) \leqq \psi_u(x) = \psi(x).$$

Hence from $$f_u(x) \leqq \psi_n(x)$$

we deduce $$f_u(x) \leqq \bar{f}(x).$$

Thus $f_u(x)$ is the least u-function equivalent to $f(x)$, and is less than any continuous function which exceeds $f(x)$.

11. **Representative for Class II.** The $f_u(x)$ just constructed serves as the smoothest representative function for any bounded function of class II. By appropriate sign changes, we may also use a function f_l, the greatest l-function equivalent to $f(x)$, which is greater than any continuous function less than $f(x)$.

If $f(x)$ is equivalent to a bounded function we may form $F(x) = \int_a^x f(x)\,dx$ and obtain an $f_u(x)$ and an $f_l(x)$ by the construction described.

If either of these is continuous, both are, and they are equal. The function $f(x)$ is then of our first class.

If either is Riemann integrable, both are, and they are equivalent to each other and to $f(x)$. The function $f(x)$ is then of our second class.

12. **Functions of type lu equivalent to summable and set functions.** Let $f(x)$ be a summable function, and consider, for any h, the function $F(x) = \frac{1}{h}\int_x^{x+h} f(x)\,dx$. For all h such that $|h| \leqq \frac{1}{N}$, and a given x, let the lower bound of these functions $F(x)$ be $F_N(x)$. As the lower bound of a set of continuous functions, $F_N(x)$ is a u-function. As N becomes infinite, these u-functions $F_N(x)$ increase (or at least are non-decreasing) and so approach an lu function which we shall call $\bar{f}_{lu}(x)$. Evidently this function is the minimum of D_+ and D_-, the lower right and left derived numbers of $\int_a^x f(x)\,dx$. Since these derived numbers are each equivalent to $f(x)$, so is $\bar{f}_{lu}(x)$.

Next consider a measurable set on a finite interval, defining a set (or characteristic) function $E(x)$, one on the set and zero on the complementary set. The metric density μ of the set at x is defined as the limit, if it exists, as h, k approach zero independently through positive values of $\frac{1}{h+k}\int_{x-h}^{x+k} E(x)\,dx$. In any case, we may form $\underline{\mu}$ and $\bar{\mu}$, respectively the lower and upper limits of metric density. Evidently $\underline{\mu}$ is the minimum of D_+ and D_- for $\int_a^x E(x)\,dx$, and hence is $\bar{E}_{lu}(x)$. The values of $\bar{E}_{lu}(x)$ satisfy

$$0 \leqq \bar{E}_{lu}(x) \leqq 1.$$

We now define a new function E_{lu}, equal to 0 if $\bar{E}_{lu}(x)$ is zero and equal to 1 if $\bar{E}_{lu}(x) > 0$. We now recall that a necessary and sufficient condition for a function to be an lu function is that for all A, the set of points for which $f(x) > A$ is the sum of an enumerable number of closed sets. Thus, since $\bar{E}_{lu}(x)$ is lu, this condition is satisfied for the points where $\bar{E}_{lu}(x) > 0$, and hence for any A for the set $E_{lu}(x) > A$, since this set is 0, the set just mentioned, or the whole interval, according as $A \geqq 1$, $0 \leqq A < 1$, or $A < 0$. Hence the function $E_{lu}(x)$ is lu, as our notation has anticipated.

Except for a set of measure zero, the metric density $\mu = 0$, or 1, a known fact which follows from the fact that $\bar{E}_{lu}(x)$ is equivalent to $E(x)$. Hence the function $E_{lu}(x)$ is equivalent to $E(x)$.

Next consider a bounded and measurable function $S(x)$, taking on only a finite number of values a_i, increasing with i, on a finite interval. For each value a_i, we form $E_i(x) = 1$ if $S(x) \geqq a_i$, and $E_i(x) = 0$ if $S(x) < a_i$. We then form $E_{ilu}(x)$ for each of these functions and $A_{ilu}(x) = a_i E_{ilu}(x)$. Finally we define $S_{lu}(x)$, for any x as the largest of the functions $A_{ilu}(x)$. It is an lu function, as the upper bound of a set of lu functions. It is equivalent to $S(x)$, since the set of points where $S(x) = a_i$ are those where $E_i(x) = 1$ and at the same

time $E_{i+1}(x) = 0$. But these two sets are equivalent to the sets where $A_{ilu}(x) = a_i$, and where $A_{(i+1)lu}(x) = 0$.

13. **Functions of type *lu* for any functions.** For any measurable function $f(x)$ we shall now define a function $f_{lu}(x)$, of type *lu*, equivalent to it on any finite interval. We begin by forming a function on any finite interval like $S(x)$, but taking on an enumerable number of values, and approximating $f(x)$ uniformly to within ϵ. We consider a set of such functions $S_n(x)$, each approximating $f(x)$ at all points where $f(x)$ is finite to within ϵ_n, with ϵ_n decreasing to zero. We take each ϵ_n a sub-multiple of the preceding, and let each set of a_i be retained in the following set. In particular, let us take $S_n(x) = a_i$ when $a_i \leqq f(x) < a_{i+1}$. Then the $S_{nlu}(x)$ formed from $S_n(x)$ form an increasing (or at least never-decreasing) sequence, approaching an *lu* function $f_{lu}(x)$.

We show that this definition is independent of the a_i used, by finding its equivalent in terms of metric density. We shall show that $f_{lu}(x_0) = a$, if for all ϵ sufficiently small, the set $f(x) > a - \epsilon$ has $\mu > 0$ at x_0, and the set $f(x) > a + \epsilon$ has $\mu = 0$ at x_0. For if $a_i \leqq a < a_{i+1}$, the set $S_n(x) \geqq a_{i+1}$ is included in $f(x) \geqq a_{i+1} > a$, and so eventually has $\mu = 0$, while the set $S_n(x) = a_{i-1}$ or a_i includes the set $a_{i-1} \leqq f(x) < a_{i+1}$ and so eventually has $\mu > 0$. Thus $S_{nlu}(x_0)$ is a_{i-1} or a_i, within $2\epsilon_n$ of a, and so approaches a, and $f_{lu}(x_0) = a$.

We see that $f_{lu}(x)$ is equivalent to $f(x)$, by noting that if we omit the set where both are infinite, $|f(x) - f_{lu}(x)|$ has its integral over any sub-interval zero.

14. **Representative functions for classes III and IV.** The function $f_{lu}(x)$ equivalent to $f(x)$ may serve as the smoothest representative function for any function of our class III or IV.

With appropriate changes in sign in the definition of $f_{lu}(x)$, we may define $f_{ul}(x)$ as a *ul* function equivalent to $f(x)$. We have $f_{ul}(x_0) = a$ if the set $f(x) > a + \epsilon$ has $\bar{\mu} < 1$, and the set $f(x) > a - \epsilon$ has $\bar{\mu} = 1$ at x_0. This may also serve as the smoothest representative function for any function of our class III or IV.

These functions $f_{lu}(x)$ and $f_{ul}(x)$, have the following properties which we state for $f_{lu}(x)$ only, in most cases, the results being similar for $f_{ul}(x)$.

a) *They are the same for any two equivalent functions.*

b) *If* $f(x) \geqq g_1(x)$, *and* $g(x)$ *is any function equivalent to* $g_1(x)$, *then* $f_{lu}(x) \geqq g_{lu}(x)$. This follows from the easy consequence of the definition that if $f(x) \geqq 0$ at all points, so is $f_{lu}(x)$.

c) *If* $\lim_{n=\infty} g_n(x) = f_1(x)$, *uniformly in* x, *where* $f_1(x)$ *is any function equivalent to* $f(x)$ *then* $\lim_{n=\infty} g_{nlu}(x) = f_{lu}(x)$ *uniformly in* x. This follows from the fact that if $|f(x)| \leqq \eta$, so is $|f_{lu}(x)| \leqq \eta$.

d) *At a point of approximate continuity of* $f(x)$, *that is where the set* $f(x) > a - \epsilon$ *has metric density* $\mu = 1$, *and the set* $f(x) > a + \epsilon$ *has metric density* $\mu = 0$, $f_{lu}(x) = a$. *Hence* $f_{lu}(x) = f_{ul}(x)$ *at these points.*

e) *If* $f(x)$ *and* $g(x)$ *are approximately continuous at* x_0, *then* $[f(x_0) + g(x_0)]_{lu} = f_{lu}(x_0) + g_{lu}(x_0)$.

Since $f(x)$ is approximately continuous at all points of the Lebesgue set, where $\lim_{h\to 0} \int_x^{x+h} |f(t) - \alpha|\, dt = |f(x) - \alpha|$ for all α, properties d) and e) apply at these points.

15. **Approximate limits.** Kempisty has defined the greatest approximate limit of a function at a point x_0 on the right L^+, as the lower bound of real numbers, a, for which the metric density of the set $f(x) \leqq a$, $x \geqq x_0$ is 1. This amounts to taking L^+ as the number dividing the class $\bar{\mu}$ of the set $f(x) > a$, $x \geqq x_0$ for $\bar{\mu} = 0$ and $\bar{\mu} > 0$. Similarly his definition of l^+ is the number dividing the class μ for the same set for $\mu < 1$ and $\mu = 1$.

He proves, as follows, that the segments $l^+ \leqq y \leqq L^+$ and $l^- \leqq y \leqq L^-$, must overlap except at an enumerable number of points. Here l^- and L^- are defined similarly to l^+ and L^+, except that the sets on the left of x_0, $x \leqq x_0$ are used. If the rationals be enumerated, r_i, for any point at which $l^- > L^+$, there is a rational r_k with least k such that $l^- > r_k > L^+$. Since $\bar{\mu} = 0$ for $f(x) > L^+$, and $x \geqq x_0$, there is a rational point r_m with least m such that for any ξ, $x_0 < \xi < r_m$, the measure of $f(x) > r_k$ in the interval $x_0 < x < \xi$ is less than $\epsilon(x - x_0)$. Similarly, since $\mu = l$ for $f(x) > l^-$, and $x \leqq x_0$, there is a rational number r_n with least n such that for any ξ, $r_n < \xi < x_0$, the measure of $f(x) > r_k$ in the interval $\xi < x < x_0$ is greater than $(1 - \epsilon)(x_0 - x)$. Hence, if we take $\epsilon = \frac{1}{2}$, we associate with each point where $l^- > L^+$, a triad of rational numbers r_k, r_m, r_n. Two different points x_0 and x_0' can not have the same triad, as this leads to contrary results for the measure of $f(x) > r_k$ in the interval $x_0 < x < x_0'$. But, the triads are enumerable.

The argument proves the further result that the points x_0 where, on one side of x_0, μ^+ is greater than $\bar{\mu}^-$ for the other side form at most an enumerable set.

If we define $(ul)^+$ as the number separating $\mu = 0$ from $\mu > 0$ on the right, and $(lu)^+$ as the number separating $\bar{\mu} < 1$ from $\bar{\mu} = 1$ on the right, for the set $f(x) > a$, we see that

$$L^+ \geqq (lu)^+ \geqq l^+ \quad \text{and} \quad L^+ \geqq (ul)^+ \geqq l^+ .$$

so that the difference $|(ul)^+ - (lu)^+| \leqq L^+ - l^+$. The difference $(ul)^+ - (lu)^+$ may have either sign, we may have, simultaneously $\bar{\mu} = 1$, $\mu = 0$. An example is obtained from the set formed on the middle third principle, but using first $\frac{1}{2}$, then $\frac{1}{3}$, then $\frac{1}{4}$, etc. The function which is 1 on the even intervals and 0 on the odd ones has $\bar{\mu} = 1$, $\mu = 0$ on one side at all the end points of intervals.

Our functions $f_{lu}(x)$ and $f_{ul}(x)$ are obtained by selecting the largest of $(lu)^+$ and $(lu)^-$; and the smallest of $(ul)^+$ and $(ul)^-$.

16. **Relations of f_{lu} and f_{ul} to f_u and f_l.** Suppose $f(x)$ is a u-function at x_0. Then, since $f(x_0 + \theta h) < f(x_0) + \epsilon$, for all sufficiently small h, for the metric density of $f(x) > f(x_0) + \epsilon$, we have $\mu = 0$. Hence $\bar{\mu} < 1$, and $\mu = 0$, so that

$$f(x_0) \geqq f_{lu}(x_0) \quad \text{and} \quad f(x_0) \geqq f_{ul}(x_0).$$

Suppose $f(x)$ is bounded in some interval. Then we may form $f_u(x)$, as in section 10. Since $f_{lu}(x)$ is equivalent to $f(x)$, and the inequality just found holds,

$$f_u(x) = [f_{lu}(x)]_u \geqq f_{lu}(x).$$

Similarly

$$f_u(x) \geqq f_{ul}(x)$$

and the same considerations applied to $f_l(x)$ show that $f_{lu}(x)$ and $f_{ul}(x)$ each lie between $f_u(x)$ and $f_l(x)$.

Since, if the function is of our class I, $f_u(x) = f_l(x)$; it follows that for these functions we shall have $f_{lu}(x)$ and $f_{ul}(x)$ reducing to $f_u(x)$ and $f_l(x)$.

Again, if the function is of our class II, since $f_u(x)$ and $f_l(x)$ are Riemann integrable, so are the functions $f_{lu}(x)$ and $f_{ul}(x)$ which lie between them.

If $f(x)$ is continuous at x_0, in the interval $x_0 - \delta < x < x_0 + \delta$ in which $|f(x) - f(x_0)| < \epsilon$, we shall have

$$f(x_0) - \epsilon < f_l(x) \leqq f_u(x) < f(x_0) + \epsilon,$$

so that $f_l(x)$ and $f_u(x)$ are both continuous at this point. Hence, or directly, we see that $f_{lu}(x)$ and $f_{ul}(x)$ are both continuous at x_0 any point of continuity of $f(x)$.

If $f(x)$ is of our class III, it is equivalent to a function of Baire's first class, $g(x)$. This function is continuous on an everywhere dense set. Hence $f_{lu}(x)$ and $f_{ul}(x)$, equal respectively to $g_{lu}(x)$ and $g_{ul}(x)$, are continuous on an everywhere dense set. Thus they are of Baire's first class.

Hence the functions $f_{lu}(x)$ and $f_{ul}(x)$ correctly represent the general function as to Baire's class, being continuous, Riemann integrable, of Baire's first class, or of Baire's second class whenever $f(x)$ admits any such equivalent function. They are thus of the smoothest possible type of equivalent function.

17. The discontinuities of $f_{lu}(x)$ and $f_{ul}(x)$. The functions $f_{lu}(x)$ and $f_{ul}(x)$ preserve continuity at all points where $f(x)$ has this property. To show to what extent they may preserve discontinuity, we shall now prove that, *given any function $f(x)$, there exists a function $g(x)$ such that $g_{lu}(x)$ or $g_{ul}(x)$ are discontinuous at all points where $f(x)$ is discontinuous.*

We begin by recalling Carathéodory's example of a function of class IV, totally discontinuous on an interval. We mark off, on a unit interval, a centered interval of length $\frac{1}{4}$. Then, in each of the remaining intervals, we mark off a centered interval of length $\frac{1}{4^2}$, and repeat the process, obtaining a set of open intervals B_1, of total length $\frac{1}{4} + 2\cdot\frac{1}{4^2} + 2^2\cdot\frac{1}{4^3} + \cdots = \frac{1}{2}$. We then define in succession: A_1, the closed set complementary to B_1 on the unit interval. A_2, the set obtained from B_1 by omitting sets of open intervals B_2, bearing the same relation to the intervals of B_2 that B_1 did to the unit interval. A_3 is obtained from B_2 by omitting sets of open intervals B_3, and so on. Now form the set $A_1 + A_3 + A_5 + \cdots$, and let $E(x)$ be 1 on this set and zero on the comple-

mentary set. On any interval of B_k, the measure of the set A_{k+1} is of measure $\frac{1}{2}$ of its length. But, as any point contains in any neighborhood complete intervals belonging to odd and even B_k, it contains points where $E(x) = 1$, and where $E(x) = 0$ of positive measure, and hence is discontinuous at this point in such a way that any equivalent function is also discontinuous.

We next form a function $C(x)$ discontinuous in this way on any closed set. Let the closed set have as its complementary open set intervals I_k, which we order according to size. We put $C(x) = 1$ on I_1, $= -1$ on I_2 and then for I_k, $+1$ unless the nearest intervals on each side with smaller subscript are both marked $+1$, in which case we use -1.

If the closed set contains any intervals, we construct on each of them a function similar to $E(x)$ defined above. Then the function is continuous on the intervals I_k, but discontinuous on any point belonging to intervals of the closed set, and also on any other points of the closed set, since these, as limit points of the open intervals will have in any neighborhood some intervals where $C(x)$ is $+1$, and some where $C(x) = -1$.

Finally, let $S_1, S_2, \cdots$ be an enumerable number of closed sets. Form functions $g_i(x)$ similar to $C(x)$ such that each function $g_i(x)$ is discontinuous on S_i, and has its oscillation ϵ_i. Take $\epsilon_{i+1} < \frac{1}{2}\epsilon_i$. Then $\sum_{j=i+1}^{\infty} \epsilon_j < \frac{1}{2}\epsilon_i$. Now put $g(x) = \sum_{i=1}^{\infty} g_i(x)$. Then at any point, x_0, not in any of the closed sets S_i, x_0 is always interior to an interval where each $g_i(x)$ is continuous, and since the series converges uniformly, $g(x)$ is continuous. But at any point of an S_i, $g_i(x)$ has an oscillation ϵ_i, and so is discontinuous, and a discontinuity, once introduced, can not be cancelled since the later oscillations are too small. Finally, since all the discontinuities come from intervals, they will persist for any equivalent functions.

Finally, we recall that, for any function, the points of discontinuity form a sum of closed sets, so that our construction for $g(x)$ proves our theorem.

18. **More dimensions.** In higher dimensions, we may continue to classify functions according as they are equivalent to continuous functions, equivalent to bounded Riemann integrable functions, functions of Baire class I, or functions of Baire class II. We may again form, from any function $f(x_i)$, two functions $f_{lu}(x_i)$ and $f_{ul}(x_i)$ such that if the sets where

$$f(x_i) > a - \epsilon \text{ has } \mu > 0, \text{ and } f(x_i) > a + \epsilon \text{ has } \mu = 0,$$

at a point x_{i0}, then $f_{lu}(x_{i0}) = a$.

Here μ refers to metric density in the number of dimensions used. Most of the properties for the one dimensional case may be extended, and in particular these functions give uniquely defined functions which represent sets of equivalent functions and are themselves of simplest type.

19. **B.a.p. Functions.** A complex function of a real variable is almost periodic if it is defined for all values of x, $-\infty < x < +\infty$, and for any given ϵ, there

is a number T_ϵ, such that any interval of length exceeding T_ϵ contains a translation number for ϵ, τ_ϵ. A translation number for ϵ, for the distance type D, is a number such that

$$D[f(x), f(x + \tau_\epsilon)] < \epsilon.$$

The type of distance D determines the type of almost periodicity. We are concerned with a distance D of Besicovitch type,

$$D_B[f(x), g(x)] = \overline{\lim_{T\to\infty}} \frac{1}{T} \int_0^T |f(x) - g(x)|^2 dx,$$

which leads to functions almost periodic in the sense of Besicovitch, or B.a.p. functions.

Besicovitch has proved that these functions are the same as the functions obtained as uniform limits of trigonometric polynomials, in the sense of the distance D_B. In particular, they have Fourier Series and are uniform limits D_B of the partial sums. He also established a Riess-Fischer Theorem, to the effect that, if a sequence of terms $a_n e^{i\lambda_n x}$ is given, with $\sum |a_n|^2$ convergent, then a B.a.p. function exists, whose Fourier series consists of the terms, $a_n e^{i\lambda_n x}$. He proves this by finding a suitable series of intervals of length $x_1, x_2, \cdots x_i \cdots$ and then defining $f(x) = s_k(x) = \sum_{n=1}^{k} a_n e^{i\lambda_n x}$ on the interval I_k given by $\sum_{n=1}^{k-1} x_n < x < \sum_{n=1}^{k} x_n$. As each x_n is determined in turn sufficiently large, we in general must, and always may, have $\lim x_n = \infty$. We consider two functions as B-equivalent if $D_B[f(x), g(x)] = 0$. If two functions are B.a.p., and have the same Fourier series, they are B-equivalent. For, we have

$$D_B[f(x), s_n(x)] < \epsilon$$

$$D_B[g(x), s_n(x)] < \epsilon$$

and hence, since D_B obeys the triangle inequality

$$D_B[f(x), g(x)] < 2\epsilon \quad \text{and hence} = 0.$$

We shall now prove

For any B.a.p. function, there exists a continuous B.a.p. function, which is B-equivalent to it.

To show this, we find the Fourier coefficients of the function, and then use the Besicovitch method to determine x_n, so that the function which is $s_k(x)$ in the k^{th} interval $\sum_{n=1}^{k-1} x_n < x < \sum_{n=1}^{k} x_n$ has these coefficients. Then we determine a series of intervals h_k, such that if M_k is the maximum of $|s_k(x)|$ and $|s_{k+1}(x)|$ in the interval $\sum_{n=1}^{k-1} x_n \leqq x \leqq \sum_{n=1}^{k} x_n$, we have $h_k < x_k$ and $\frac{h_k M_k^2}{x_k} < \epsilon_k$ where the ϵ_k are positive and decrease to zero.

Finally we redefine $f(x)$ in the interval I_k' given by

$$\sum_{n=1}^{k} x_n - h_k \leqq x \leqq \sum_{n=1}^{k} x_n$$

by a straight line function, joining continuously with $s_k(x)$ and $s_{k+1}(x)$ at the end points, calling the new function $F(x)$.

Let us consider the average of $|F(x) - f(x)|^2$ in the interval $0 < x < T$, including the first x_j $(j = 1, 2, \cdots m)$, m in number. We have

$$F(x) - f(x) = 0 \quad \text{in } I_k - I_k', \text{ of length } x_k - h_k$$

$$F(x) - f(x) \leqq 2M_k \text{ in } I_k', \text{ of length } h_k.$$

From this,

$$\frac{1}{T}\int_0^T |F(x) - f(x)|^2\,dx \leqq \frac{1}{T}\sum_{k=1}^{m} 4M_k^2 h_k$$

$$\leqq \frac{4}{T}\sum_{k=1}^{m} \epsilon_k x_k$$

We next divide this sum into one of $q - 1$ terms and one of $m - q + 1$ terms, and recalling that the ϵ_k decrease, and the x_k are positive, we have

$$\sum_{k=1}^{m} \epsilon_k x_k = \sum_{k=1}^{q-1} \epsilon_k x_k + \sum_{k=q}^{m} \epsilon_k x_k$$

$$\leqq \epsilon_1 \sum_{k=1}^{q-1} x_k + \epsilon_q \sum_{k=q}^{m} x_k$$

$$\leqq \epsilon_1 \sum_{1}^{q-1} x_k + \epsilon_q T.$$

This shows that

$$\frac{1}{T}\int_0^T |F(x) - f(x)|^2\,dx \leqq 4\epsilon_1 \frac{\sum_{1}^{q-1} x_k}{T} + 4\epsilon_q.$$

If we first select q so that $4\epsilon_q < \delta$ and then keep q fixed, $4\epsilon_1 \sum_{1}^{q-1} x_k$ will remain fixed, and if we take T large enough, $T > T_\delta$, we will have

$$\frac{4\epsilon_1 \sum_{1}^{q-1} x_k}{T} < \delta.$$

Thus, we will have

$$\frac{1}{T}\int_0^T |F(x) - f(x)|^2\,dx \leqq 2\delta; \qquad T > T_\delta.$$

This proves that $F(x)$ is B-equivalent to $f(x)$. Since $f(x)$ is B.a.p., so is $F(x)$ and it has the same Fourier series.

COROLLARY. *The B-equivalent B.a.p. function may be taken to have derivatives of all orders.*

We have merely to replace the straight line parts of $F(x)$ by suitably modified arcs.

20. **Convergent Sequences of B.a.p. functions.** A sequence of B.a.p. functions will be defined as B-convergent if the relation

$$D_B[\varphi_k, \varphi_j] \leqq \epsilon \text{ for } k, j > N_\epsilon$$

is true for any ϵ for a suitable N_ϵ.

We wish to prove the

THEOREM. *If a series of B.a.p. functions is B-convergent, there is a B.a.p. function ψ to which it is B-convergent, i.e.*

$$\lim_{k\to\infty} D_B[\varphi_k, \psi] = 0.$$

Consider the set of functions φ_k. For each of them we have an enumerable set of Fourier coefficients and exponents. Let λ_n be an enumerable set of exponents including the exponents with non-zero coefficients for any of the φ_k. Then, by including terms with zero coefficients if necessary, we may consider each φ_k to have the Fourier terms $a_{nk}e^{\lambda_n x}$. Hence, except for some zero terms, the function $\varphi_k - \varphi_j$ will have the Fourier coefficients $(a_{nk} - a_{nj})$, and

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |\varphi_k - \varphi_j|^2\, dx = \sum_{n=1}^{\infty} |a_{nk} - a_{nj}|^2.$$

But, since the sequence is B-convergent for $k, j > N_\epsilon$, the left member is at most ϵ, so that

$$\sum_{n=1}^{\infty} |a_{nk} - a_{nj}|^2 < \epsilon \quad \text{for} \quad k, j > N_\epsilon.$$

Thus, the same inequality holds for any one term of the series, so that by the Cauchy criterion we have for each n a set of numbers a_{nk} which converge,

$$\lim_{k\to\infty} a_{nk} = a_n \quad \text{uniformly in } n.$$

Again

$$\sum_{n=1}^{\infty} |a_{nk}|^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T |\varphi_k|^2\, dx,$$

and

$$\sqrt{\sum |a_{nk}|^2} \leqq \sqrt{\sum |a_{nj}|^2} + \sqrt{\sum |a_{nk} - a_{nj}|^2},$$

where the sums are finite, $n = 1$ to m, or infinite.

Thus, by the Cauchy criterion, $\sum_{n=1}^{\infty} |a_{nk}|^2$ converges to a sum, say B, and we have

$$\lim_{k\to\infty} \lim_{m\to\infty} \sum_{n=1}^{m} |a_{nk}|^2 = B.$$

As the sum $\sum_{n=1}^{m} |a_{nk} - a_{nj}|^2$ increases with m, we obtain by the same Cauchy

argument that $\lim_{k\to\infty} \sum_{n=1}^{m} |a_{nk}|^2$ exists, uniformly in m, and it follows from the last two relations that we may invert the order of the limits and

$$\lim_{m\to\infty} \lim_{k\to\infty} \sum_{n=1}^{m} |a_{nk}|^2 = \sum_{n=1} |a_n|^2 = B.$$

Thus the numbers a_n may be used with the exponents λ_n to give a B.a.p. function by the theorem of Besicovitch. Let ψ be one such function. Then the Fourier series for the function $\psi - \varphi_k$ is, except for zero terms, $\sum(a_n - a_{nk})e^{\lambda_n x}$, so that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\psi - \varphi_k|^2 \, dx = \sum_{n=1}^{\infty} |a_n - a_{nk}|^2.$$

Now apply the argument previously used to the set of functions $\psi - \varphi_k$. These are B-convergent, and hence, as before

$$\lim_{k\to\infty} \lim_{m\to\infty} \sum_{n=1}^{m} |a_n - a_{nk}|^2 = \lim_{m\to\infty} \lim_{k\to\infty} \sum_{n=1}^{m} |a_n - a_{nk}|^2 = 0.$$

The last two equations show that the given sequence converges to ψ, since

$$\lim_{k\to\infty} \lim_{T\to\infty} \frac{1}{T} \int_0^T |\psi - \varphi_k|^2 \, dx = 0,$$

or

$$\lim_{k\to\infty} D_B[\psi, \varphi_k] = 0.$$

The function ψ may be taken as continuous, by the preceding section.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

TABLE OF $f_n(x) = \dfrac{n!}{(x/2)^n} J_n(x)$

By The Mathematical Tables Project

The best existing table of the function $J_n(x)$ consists of an unpublished manuscript which has been made available to the Mathematical Tables Project by the British Association for the Advancement of Science. That manuscript has the following scope:[1]

$x = 0(.1)25;\quad n = 0(1)20;\quad 10D \qquad x = 0(.01)10;\quad n = 0(1)12;\quad 8D$

Over a considerable range, the tables have but one or two significant figures, however. Since the function $J_n(x)$ is often required for a constant number of significant figures rather than a fixed number of decimal places, the table of the function

$$f_n(x) = \frac{n!}{(x/2)^n} J_n(x) = \sum_{k=0}^{\infty} (-1)^k (x/2)^{2k} \frac{n!}{k!(n+k)!} \tag{1}$$

is being made available in the following pages, for $x \leq 10$, and $n \leq 20$. The functions $f_0(x)$ and $f_1(x)$ are not given here because $J_0(x)$ and $J_1(x)$ have been adequately tabulated in this range of x.

Modified second central differences are tabulated alongside the entries.[2]

Method of Computation

The following recurrence relations hold for $f_n(x)$. (Where no ambiguity is likely to arise, these functions will be denoted by f_n respectively for the sake of brevity.)

$$f_{n-1} = f_n - \frac{x^2}{4n(n+1)} f_{n+1} \tag{2}$$

If we begin with $n = 20$ and proceed backward, no accuracy is lost until

$$\frac{x^2}{4n(n+1)} > 1; \qquad \text{or } x > 2n \text{ approximately.}$$

Thus for $x \leq 10$, no accuracy is lost for the range of n from 20 to 3. The computation therefore proceeded as follows:

The functions f_{20}, f_{19}, and f_{18} were computed from the series (1) to 10 decimal

[1] The symbol 0(.1)25; 10D, for example, indicates the range of x from 0 to 25 at intervals of 0.1 to 10 decimals.

[2] See the discussion of modified differences in British Association for the Advancement of Science. "Mathematical Tables, Vol. I", pp. xi, xii. For purposes of interpolation modified differences are used exactly like ordinary differences.

places, and checked by the recurrence relation (2). All other values of f_n, up to $n = 3$, were computed from (2), and checked by the formula

$$f_{n-1} = \left[1 - \frac{x^2}{4n(n+1)}\right] f_{n+1} - \frac{x^2}{4n(n+1)(n+2)} f_{n+2} \qquad (3)$$

This process will be referred to as the first recurrence chain. Since $f_0(x) = J_0(x)$ and $f_1(x) = \frac{2}{x} J_1(x)$, the values of f_0, f_1 were available from the tabulated values of $J_0(x)$ and $J_1(x)$. With these values of f_0 and f_1, f_2 and f_3 were computed and the latter was checked against the value of f_3 previously obtained in the first recurrence chain. Agreement could be regarded as a check on the accuracy of the entire chain of values of $f_n(x)$. For small values of x, the first recurrence chain was carried through completely, and checked against the known values of $f_1(x)$.

The final manuscript was completely differenced.

$$f_n(x) = \frac{n!}{(x/2)^n} J_n(x)$$

x	f_2	$*\delta^2$	f_3	$*\delta^2$	x	f_2	$*\delta^2$	f_3	$*\delta^2$
0.00	1.00000 0000	−416706	1.00000 0000	−312526	2.50	0.57095 5595	−41142	0.66539 6401	−76885
0.05	0.99979 1683	416511	0.99984 3759	312404	2.55	.55707 4435	30124	.65402 6289	69522
0.10	.99916 6927	415925	.99937 5156	312055	2.60	.54316 3138	19177	.64258 6654	62166
0.15	.99812 6318	414949	.99859 4541	311470	2.65	.52923 2649	−8313	.63108 4851	54822
0.20	.99667 0831	413586	.99750 2499	310652	2.70	.51529 3830	+2461	.61952 8222	47499
0.25	0.99480 1829	−411836	0.99609 9848	−309599	2.75	0.50135 7452	13126	0.60792 4089	−40204
0.30	.99252 1062	409700	.99438 7640	308317	2.80	.48743 4179	23678	.59627 9746	32943
0.35	.98983 0665	407181	.99236 7157	306805	2.85	.47353 4560	34098	.58460 2453	25720
0.40	.98673 3156	404288	.99003 9911	305065	2.90	.45966 9014	44384	.57289 9431	18546
0.45	.98323 1428	401017	.98740 7642	303096	2.95	.44584 7824	54516	.56117 7853	11424
0.50	0.97932 8751	−397376	0.98447 2318	−300905	3.00	0.43208 1121	64492	0.54944 4840	−4363
0.55	.97502 8766	393365	.98123 6130	298492	3.05	.41837 8878	74294	.53770 7452	+2636
0.60	.97033 5482	388997	.97770 1491	295857	3.10	.40475 0896	83919	.52597 2686	9559
0.65	.96525 3267	384269	.97387 1035	293004	3.15	.39120 6798	93353	.51424 7465	16409
0.70	.95978 6847	379196	.96974 7614	289940	3.20	.37775 6016	102586	.50253 8637	23171
0.75	0.95394 1295	−373774	0.96533 4292	−286664	3.25	0.36440 7782	111612	0.49085 2964	29846
0.80	.94772 2031	368017	.96063 4344	283181	3.30	.35117 1120	120423	.47919 7120	36429
0.85	.94113 4811	361929	.95565 1253	279493	3.35	.33805 4839	129007	.46757 7686	42908
0.90	.93418 5721	355520	.95038 8706	275604	3.40	.32506 7522	137357	.45600 1141	49284
0.95	.92688 1169	348794	.94485 0591	271520	3.45	.31221 7518	145469	.44447 3860	55551
1.00	0.91922 7879	−341765	0.93904 0991	−267244	3.50	0.29951 2937	153331	0.43300 2108	61699
1.05	.91123 2879	334435	.93296 4181	262782	3.55	.28696 1640	160939	.42159 2033	67729
1.10	.90290 3497	326820	.92662 4623	258133	3.60	.27457 1234	168286	.41024 9664	73636
1.15	.89424 7347	318922	.92002 6964	253311	3.65	.26234 9065	175364	.39898 0907	79413
1.20	.88527 2324	310757	.91317 6026	248312	3.70	.25030 2210	182171	.38779 1538	85055
1.25	0.87598 6592	−302332	0.90607 6807	−243146	3.75	0.23843 7475	188702	0.37668 7199	90560
1.30	.86639 8574	293657	.89873 4472	237817	3.80	.22676 1389	194945	.36567 3394	95926
1.35	.85651 6943	284744	.89115 4349	232331	3.85	.21528 0195	200901	.35475 5488	101145
1.40	.84635 0610	275603	.88334 1923	226690	3.90	.20399 9849	206569	.34393 8700	106216
1.45	.83590 8714	266243	.87530 2833	220908	3.95	.19292 6018	211941	.33322 8100	111134
1.50	0.82520 0612	−256683	0.86704 2861	−214983	4.00	0.18206 4073	217014	0.32262 8606	115899
1.55	.81423 5863	246924	.85856 7931	208922	4.05	.17141 9086	221783	.31214 4982	120506
1.60	.80302 4223	236987	.84988 4102	202735	4.10	.16099 5827	226253	.30178 1834	124950
1.65	.79157 5628	226876	.84099 7560	196427	4.15	.15079 8765	230418	.29154 3606	129234
1.70	.77990 0186	216610	.83191 4613	190000	4.20	.14083 2064	234272	.28143 4581	133351
1.75	0.76800 8161	−206196	0.82264 1686	−183463	4.25	0.13109 9579	237820	0.27145 8876	137299
1.80	.75590 9964	195651	.81318 5314	176828	4.30	.12160 4858	241062	.26162 0439	141080
1.85	.74361 6138	184985	.80355 2132	170094	4.35	.11235 1142	243994	.25192 3050	144687
1.90	.73113 7347	174209	.79374 8873	163269	4.40	.10334 1363	246616	.24237 0316	148123
1.95	.71848 4364	163339	.78378 2360	156364	4.45	.09457 8143	248929	.23296 5673	151383
2.00	0.70566 8057	−152386	0.77365 9497	−149381	4.50	0.08606 3796	250939	0.22371 2381	154471
2.05	.69269 9377	141365	.76338 7266	142327	4.55	.07780 0332	252642	.21461 3527	157381
2.10	.67958 9343	130285	.75297 2719	135214	4.60	.06978 9454	254039	.20567 2021	160114
2.15	.66634 9032	119161	.74242 2968	128044	4.65	.06203 2560	255138	.19689 0596	162668
2.20	.65298 9566	108006	.73174 5182	120827	4.70	.05453 0749	255935	.18827 1807	165048
2.25	0.63952 2097	− 96834	0.72094 6577	−113565	4.75	0.04728 4819	256438	0.17981 8033	167246
2.30	.62595 7795	85656	.71003 4413	106271	4.80	.04029 5273	256650	.17153 1473	169272
2.35	.61230 7836	74484	.69901 5983	98950	4.85	.03356 2323	256568	.16341 4152	171116
2.40	.59858 3389	63333	.68789 8607	91605	4.90	.02708 5889	256205	.15546 7915	172785
2.45	.58479 5603	52213	.67668 9628	84249	4.95	.02086 5608	255561	.14769 4431	174278
2.50	0.57095 5595	−41142	0.66539 6401	−76885	5.00	0.01490 0837	254641	0.14009 5193	175596

* These differences are modified.

$$f_n(x) = \frac{n!}{(x/2)^n} J_n(x)$$

x	f_2	$*\delta^2$	f_3	$*\delta^2$	x	f_2	$*\delta^2$	f_3	$*\delta^2$
5.00	0.01490 0837	254641	0.14009 5193	175596	7.50	−0.03274 9996	20693	−0.02936 1597	79728
5.05	.00919 0657	253449	.13267 1519	176738	7.55	.03090 5351	15881	.02945 7838	76306
5.10	+0.00373 3877	251991	.12542 4552	177711	7.60	.02904 4803	11189	.02947 7768	72906
5.15	−0.00147 0959	250277	.11835 5264	178508	7.65	.02717 3044	6617	.02942 4789	69529
5.20	.00642 5565	248306	.11146 4453	179139	7.70	.02529 4644	+2177	.02930 2286	66175
5.25	−0.01113 1910	246090	0.10475 2750	179602	7.75	−0.02341 4043	−2131	−0.02911 3622	62846
5.30	.01559 2210	243630	.09822 0618	179895	7.80	.02153 5548	6305	.02886 2136	59538
5.35	.01980 8923	240938	.09186 8351	180027	7.85	.01966 3332	10339	.02855 1136	56229
5.40	.02378 4740	238017	.08569 6081	179998	7.90	.01780 1428	14226	.02818 3900	52980
5.45	.02752 2580	234877	.07970 3779	179807	7.95	.01595 3724	17973	.02776 3668	49784
5.50	−0.03102 5582	231525	0.07389 1255	179461	8.00	−0.01412 3965	−21567	−0.02729 3644	46623
5.55	.03429 7097	227967	.06825 8164	178963	8.05	.01231 5745	25011	.02677 6989	43506
5.60	.03734 0681	224213	.06280 4008	178310	8.10	.01053 2508	28302	.02621 6819	40437
5.65	.04016 0087	220268	.05752 8135	177512	8.15	.00877 7545	31440	.02561 6203	37416
5.70	.04275 9258	216144	.05242 9747	176568	8.20	.00705 3993	34419	.02497 8161	34451
5.75	−0.04514 2317	211847	0.04750 7901	175484	8.25	−0.00536 4831	−37240	−0.02430 5658	31537
5.80	.04731 3559	207387	.04276 1513	174260	8.30	.00371 2880	39903	.02360 1607	28684
5.85	.04927 7443	202770	.03818 9360	172902	8.35	.00210 0803	42407	.02286 8861	25891
5.90	.05103 8584	198005	.03379 0085	171412	8.40	−0.00053 1104	44751	.02211 0213	23157
5.95	.05260 1745	193106	.02956 2199	169796	8.45	+0.00099 3873	46936	.02132 8396	20489
6.00	−0.05397 1824	188075	0.02550 4086	168056	8.50	0.00247 1944	−48958	−0.02052 6078	17886
6.05	.05515 3850	182924	.02161 4007	166199	8.55	.00390 1086	50824	.01970 5862	15349
6.10	.05615 2972	177664	.01789 0105	164222	8.60	.00527 9434	52527	.01887 0284	12884
6.15	.05697 4449	172301	.01433 0405	162136	8.65	.00660 5284	54074	.01802 1809	10488
6.20	.05762 3642	166844	.01093 2821	159942	8.70	.00787 7089	55465	.01716 2833	8164
6.25	−0.05810 6006	161304	0.00769 5160	157647	8.75	0.00909 3458	−56700	−0.01629 5680	5911
6.30	.05842 7080	155689	.00461 5127	155249	8.80	.01025 3156	57778	.01542 2602	3737
6.35	.05859 2477	150008	+0.00169 0326	152761	8.85	.01135 5104	58705	.01454 5774	+1634
6.40	.05860 7876	144271	−0.00108 1731	150180	8.90	.01239 8374	59486	.01366 7298	−392
6.45	.05847 9013	138485	.00370 3624	147514	8.95	.01338 2186	60115	.01278 9200	2340
6.50	−0.05821 1672	132662	−0.00617 8018	144767	9.00	0.01430 5910	−60598	−0.01191 3428	−4214
6.55	.05781 1675	126808	.00850 7659	141943	9.05	.01516 9062	60941	.01104 1855	6005
6.60	.05728 4874	120931	.01069 5370	139049	9.10	.01597 1299	61142	.01017 6273	7722
6.65	.05663 7144	115043	.01274 4045	136085	9.15	.01671 2419	61207	.00931 8398	9357
6.70	.05587 4372	109150	.01465 6647	133056	9.20	.01739 2356	61140	.00846 9866	10918
6.75	−0.05500 2449	103264	−0.01643 6203	129973	9.25	0.01801 1177	−60943	−0.00763 2237	−12398
6.80	.05402 7260	97387	.01808 5796	126834	9.30	.01856 9079	60615	.00680 6990	13798
6.85	.05295 4680	91531	.01960 8564	123646	9.35	.01906 6388	60168	.00599 5526	15119
6.90	.05179 0563	85707	.02100 7694	120412	9.40	.01950 3551	59602	.00519 9167	16364
6.95	.05054 0732	79918	.02228 6419	117139	9.45	.01988 1133	58922	.00441 9158	17532
7.00	−0.04921 0975	74173	−0.02344 8012	113828	9.50	0.02019 9814	−58127	−0.00365 6667	−18622
7.05	.04780 7036	68478	.02449 5782	110490	9.55	.02046 0387	57230	.00291 2784	19633
7.10	.04633 4608	62844	.02543 3067	107121	9.60	.02066 3749	56229	.00218 8521	20573
7.15	.04479 9324	57277	.02626 3235	103730	9.65	.02081 0900	55129	.00148 4817	21434
7.20	.04320 6750	51779	.02698 9676	100323	9.70	.02090 2939	53940	.00080 2534	22224
7.25	−0.04156 2382	46366	−0.02761 5797	96901	9.75	0.02094 1055	−52657	−0.00014 2461	−22939
7.30	.03987 1633	41034	.02814 5019	93469	9.80	.02092 6529	51294	+0.00049 4687	23583
7.35	.03813 9833	35797	.02858 0773	90031	9.85	.02086 0724	49849	.00110 8266	24155
7.40	.03637 2218	30657	.02892 6496	86592	9.90	.02074 5083	48333	.00169 7703	24659
7.45	.03457 3927	25620	.02918 5626	83158	9.95	.02058 1122	46747	.00226 2496	25091
7.50	−0.03274 9996	20693	−0.02936 1597	79728	10.00	0.02037 0425	−45102	0.00280 2210	−25460

* These differences are modified.

$$f_n(x) = \frac{n!}{(x/2)^n} J_n(x)$$

x	f_4	$*\delta^2$	f_5	$*\delta^2$	x	f_4	$*\delta^2$	f_5	$*\delta^2$
.00	1.00000 0000	−250017	1.00000 0000	−208345	2.75	0.67638 9894	−61374	0.72426 6373	−69705
.05	0.99987 5006	249936	0.99989 5838	208288	2.80	.66640 1429	56067	.71552 7382	65651
.10	.99950 0104	249704	.99958 3408	208121	2.85	.65635 6896	50765	.70672 2741	61591
.15	.99887 5527	249312	.99906 2877	207844	2.90	.64626 1596	45475	.69785 6509	57531
.20	.99800 1666	248768	.99833 4523	207451	2.95	.63612 0818	40201	.68893 2746	53468
.25	0.99687 9066	−248067	0.99739 8738	−206953	3.00	0.62593 9835	−34948	0.67995 5514	−49409
.30	.99550 8428	247210	.99625 6021	206340	3.05	.61572 3900	29717	.67092 8872	45356
.35	.99389 0608	246200	.99490 6984	205618	3.10	.60547 8243	24514	.66185 6873	41308
.40	.99202 6616	245041	.99335 2349	204788	3.15	.59520 8066	19345	.65274 3564	37274
.45	.98991 7612	243724	.99159 2946	203848	3.20	.58491 8538	14208	.64359 2979	33251
.50	0.98756 4911	−242260	0.98962 9715	−202800	3.25	0.57461 4795	−9111	0.63440 9140	−29246
.55	.98496 9977	240648	.98746 3704	201644	3.30	.56430 1933	−4060	.62519 6052	25257
.60	.98213 4423	238883	.98509 6068	200383	3.35	.55398 5003	+947	.61595 7703	21293
.65	.97906 0012	236977	.98252 8068	199016	3.40	.54366 9011	5903	.60669 8057	17349
.70	.97574 8651	234925	.97976 1071	197545	3.45	.53335 8912	10807	.59742 1057	13433
.75	0.97220 2392	−232726	0.97679 6548	−195971	3.50	0.52305 9609	15651	0.58813 0619	− 9545
.80	.96842 3432	230392	.97363 6073	194293	3.55	.51277 5946	20434	.57883 0630	5689
.85	.96441 4106	227916	.97028 1323	192518	3.60	.50251 2706	25157	.56952 4946	− 1868
.90	.96017 6889	225305	.96673 4074	190640	3.65	.49227 4610	29809	.56021 7388	+ 1919
.95	.95571 4392	222557	.96299 6203	188666	3.70	.48206 6310	34394	.55091 1742	5667
1.00	0.95102 9362	−219680	0.95906 9684	−186596	3.75	0.47189 2390	38904	0.54161 1756	9377
1.05	.94612 4676	216671	.95495 6587	184429	3.80	.46175 7360	43336	.53232 1139	13041
1.10	.94100 3342	213539	.95065 9078	182172	3.85	.45166 5652	47692	.52304 3555	16660
1.15	.93566 8492	210279	.94617 9414	179823	3.90	.44162 1621	51965	.51378 2623	20235
1.20	.93012 3385	206901	.94151 9944	177382	3.95	.43162 9540	56155	.50454 1917	23758
1.25	0.92437 1399	−203403	0.93668 3108	−174854	4.00	0.42169 3598	60255	0.49532 4960	27232
1.30	.91841 6031	199791	.93167 1433	172244	4.05	.41181 7895	64268	.48613 5225	30653
1.35	.91226 0893	196066	.92648 7530	169547	4.10	.40200 6443	68191	.47697 6133	34017
1.40	.90590 9709	192233	.92113 4095	166767	4.15	.39226 3164	72017	.46785 1048	37325
1.45	.89936 6311	188295	.91561 3907	163911	4.20	.38259 1884	75747	.45876 3278	40579
1.50	0.89263 4637	−184254	0.90992 9822	−160977	4.25	0.37299 6333	79382	0.44971 6075	43767
1.55	.88571 8727	180117	.90408 4774	157965	4.30	.36348 0145	82912	.44071 2628	46895
1.60	.87862 2718	175882	.89808 1774	154883	4.35	.35404 6851	86345	.43175 6065	49962
1.65	.87135 0844	171556	.89192 3904	151731	4.40	.34469 9883	89671	.42284 9452	52962
1.70	.86390 7430	167146	.88561 4316	148508	4.45	.33544 2567	92896	.41399 5789	55897
1.75	0.85629 6886	−162649	0.87915 6232	−145221	4.50	0.32627 8127	96011	0.40519 8010	58762
1.80	.84852 3708	158073	.87255 2939	141870	4.55	.31720 9678	99019	.39645 8981	61562
1.85	.84059 2471	153420	.86580 7787	138460	4.60	.30824 0228	101919	.38778 1501	64289
1.90	.83250 7827	148698	.85892 4186	134989	4.65	.29937 2677	104709	.37916 8297	66947
1.95	.82427 4498	143904	.85190 5606	131465	4.70	.29060 9814	107385	.37062 2026	69529
2.00	0.81589 7276	−139050	0.84475 5571	−127886	4.75	0.28195 4316	109953	0.36214 5271	72042
2.05	.80738 1015	134133	.83747 7659	124258	4.80	.27340 8750	112406	.35374 0544	74478
2.10	.79873 0631	129161	.83007 5498	120580	4.85	.26497 5569	114745	.34541 0281	76838
2.15	.78995 1095	124139	.82255 2765	116861	4.90	.25665 7112	116972	.33715 6842	79124
2.20	.78104 7429	119068	.81491 3179	113098	4.95	.24845 5606	119083	.32898 2513	81334
2.25	0.77202 4703	−113956	0.80716 0503	−109292	5.00	0.24037 3162	121080	0.32088 9503	83463
2.30	.76288 8029	108801	.79929 8541	105453	5.05	.23241 1777	122962	.31287 9942	85518
2.35	.75364 2560	103615	.79133 1132	101581	5.10	.22457 3333	124729	.30495 5884	87492
2.40	.74429 3482	98397	.78326 2148	97675	5.15	.21685 9597	126381	.29711 9303	89387
2.45	.73484 6012	93152	.77509 5494	93742	5.20	.20927 2221	127918	.28937 2094	91203
2.50	0.72530 5394	− 87888	0.76683 5103	− 89783	5.25	0.20181 2742	129340	0.28171 6073	92940
2.55	.71567 6892	82601	.75848 4933	85800	5.30	.19448 2582	130650	.27415 2977	94596
2.60	.70596 5791	77306	.75004 8966	81799	5.35	.18728 3051	131844	.26668 4462	96172
2.65	.69617 7386	71997	.74153 1203	77782	5.40	.18021 5343	132924	.25931 2104	97668
2.70	.68631 6984	66687	.73293 5661	73748	5.45	.17328 0539	133895	.25203 7399	99084
2.75	0.67638 9894	− 61374	0.72426 6373	− 69705	5.50	0.16647 9609	134752	0.24486 1763	100417

*These differences are modified.

$$f_n(x) = \frac{n!}{(x/2)^n} J_n(x)$$

x	f_4	$*\delta^2$	f_5	$*\delta^2$
5.50	0.16647 9609	134752	0.24486 1763	100417
5.55	.15981 3410	135496	.23778 6530	101674
5.60	.15328 2687	136135	.23081 2956	102850
5.65	.14688 8078	136659	.22394 2217	103945
5.70	.14063 0109	137081	.21717 5408	104958
5.75	0.13450 9201	137393	0.21051 3543	105897
5.80	.12852 5667	137601	.20395 7560	106755
5.85	.12267 9715	137703	.19750 8317	107533
5.90	.11697 1448	137706	.19116 6593	108235
5.95	.11140 0869	137608	.18493 3090	108859
6.00	0.10596 7880	137409	0.17880 8432	109408
6.05	.10067 2282	137112	.17279 3168	109878
6.10	.09551 3779	136724	.16688 7769	110278
6.15	.09049 1982	136236	.16109 2634	110598
6.20	.08560 6405	135662	.15540 8084	110850
6.25	0.08085 6473	134995	0.14983 4370	111025
6.30	.07624 1520	134240	.14437 1668	111131
6.35	.07176 0792	133403	.13902 0084	111166
6.40	.06741 3451	132477	.13377 9653	111133
6.45	.06319 8573	131476	.12865 0342	111028
6.50	.05911 5156	130392	0.12363 2047	110859
6.5	0.05911 5156	521660	0.12363 2047	443515
6.6	.05133 8296	512082	.11392 7762	441366
6.7	.04407 3307	501357	.10466 4656	438203
6.8	.03730 9478	489561	.09583 9575	434066
6.9	.03103 5028	476776	.08744 8388	429001
7.0	0.02523 7188	463085	0.07948 6041	423055
7.1	.01990 2281	448571	.07194 6595	416276
7.2	.01501 5809	433317	.06482 3280	408712
7.3	.01056 2535	417409	.05810 8543	400417
7.4	.00652 6564	400930	.05179 4098	391440
7.5	+0.00289 1434	383965	0.04587 0977	381837
7.6	− .00035 9806	366597	.04032 9589	371658
7.7	.00324 4507	348907	.03515 9762	360959
7.8	.00578 0346	330977	.03035 0809	349793
7.9	.00798 5237	312885	.02589 1572	338212
8.0	−0.00987 7260	294708	0.02177 0481	326272
8.1	.01147 4576	276520	.01797 5605	314024
8.2	.01279 5360	258393	.01449 4706	301520
8.3	.01385 7729	240395	.01131 5289	288813
8.4	.01467 9666	222593	.00842 4657	275950
8.5	−0.01527 8893	205047	0.00580 9955	262983
8.6	.01567 3154	187819	.00345 8226	249959
8.7	.01587 9459	170960	+ .00135 6453	236921
8.8	.01591 4726	154525	− .00050 8393	223917
8.9	.01579 5382	138558	.00214 9309	210986
9.0	−0.01553 7386	123106	−0.00357 9217	198172
9.1	.01515 6182	108204	.00481 0926	185510
9.2	.01466 6667	93892	.00585 7089	173040
9.3	.01408 3146	80195	.00673 0171	160793
9.4	.01341 9313	67144	.00744 2414	148802
9.5	−0.01268 8212	54761	−0.00800 5801	137098
9.6	.01190 2224	43063	.00843 2034	125706
9.7	.01107 3044	32067	.00873 2499	114652
9.8	.01021 1666	21780	.00891 8245	103958
9.9	.00932 8377	12212	.00899 9963	93646
10.0	−0.00843 2743	3364	−0.00898 7963	83731

* These differences are modified.

$$f_n(x) = \frac{n!}{(x/2)^n} J_n(x)$$

x	f_6	$*\delta^2$	f_7	$*\delta^2$	f_8	$*\delta^2$
0.00	1.00000 0000	−178679	1.00000 0000	−156255	1.00000 0000	−138893
0.05	0.99991 0718	178539	0.99992 1878	156225	0.99993 0558	138869
0.10	.99964 2913	178412	.99968 7543	156126	.99972 2257	138789
0.15	.99919 6711	178203	.99929 7094	155963	.99937 5176	138661
0.20	.99857 2321	177912	.99875 0694	155736	.99888 9444	138477
0.25	.99777 0035	177534	.99804 8570	155445	.99826 5244	138244
0.30	.99679 0230	177076	.99719 1013	155087	.99750 2810	137957
0.35	.99563 3364	176534	.99617 8381	154664	.99660 2428	137622
0.40	.99429 9979	175913	.99501 1096	154181	.99556 4434	137232
0.45	.99279 0697	175205	.99368 9642	153629	.99438 9217	136794
0.50	0.99110 6225	−174419	0.99221 4570	−153019	0.99307 7216	−136302
0.55	.98924 7349	173549	.99058 6491	152341	.99162 8922	135760
0.60	.98721 4938	172603	.98880 6082	151604	.99004 4877	135169
0.65	.98500 9939	171576	.98687 4080	150803	.98832 5672	134528
0.70	.98263 3379	170468	.98479 1286	149940	.98647 1948	133837
0.75	.98008 6365	169286	.98255 8563	149019	.98448 4396	133096
0.80	.97737 0080	168022	.98017 6833	148033	.98236 3757	132307
0.85	.97448 5787	166684	.97764 7081	146987	.98011 0820	131471
0.90	.97143 4824	165271	.97497 0352	145886	.97772 6421	130585
0.95	.96821 8604	163783	.97214 7748	144724	.97521 1446	129651
1.00	0.96483 8615	−162220	0.96918 0431	−143504	0.97256 6828	−128673
1.05	.96129 6419	160586	.96606 9621	142225	.96979 3546	127646
1.10	.95759 3650	158882	.96281 6596	140892	.96689 2626	126574
1.15	.95373 2012	157106	.95942 2689	139504	.96386 5140	125458
1.20	.94971 3281	155262	.95588 9288	138058	.96071 2204	124297
1.25	.94553 9301	153349	.95221 7838	136563	.95743 4979	123091
1.30	.94121 1984	151371	.94840 9835	135010	.95403 4671	121844
1.35	.93673 3308	149326	.94446 6831	133409	.95051 2527	120551
1.40	.93210 5317	147221	.94039 0428	131754	.94686 9839	119221
1.45	.92733 0117	145051	.93618 2280	130052	.94310 7938	117846
1.50	.92240 9877	142821	.93184 4089	128298	.93922 8198	116432
1.5	0.92240 9877	−571356	0.93184 4089	−513254	0.93922 8198	−465778
1.6	.91214 3255	552808	.92278 4636	498662	.93112 0895	453995
1.7	.90132 3989	533363	.91322 6657	483339	.92255 9709	441606
1.8	.88997 1516	513073	.90318 5466	467324	.91355 7023	428640
1.9	.87810 6116	491990	.89267 7072	450653	.90412 5799	415122
2.0	.86574 8860	470175	.88171 8137	433369	.89427 9548	401086
2.1	.85292 1554	447679	.87032 5940	415509	.88403 2302	386558
2.2	.83964 6683	424567	.85851 8331	397121	.87339 8582	371573
2.3	.82594 7347	400897	.84631 3691	378244	.86239 3367	356162
2.4	.81184 7207	376729	.83373 0888	358924	.85103 2061	340358
2.5	0.79737 0417	−352127	0.82078 9235	−339204	0.83933 0464	−324195
2.6	.78254 1568	327154	.80750 8443	319133	.82730 4731	307706
2.7	.76738 5622	301873	.79390 8574	298758	.81497 1346	290927
2.8	.75192 7848	276347	.78000 9996	278117	.80234 7081	273892
2.9	.73619 3761	250639	.76583 3341	257265	.78944 8964	256636
3.0	.72020 9056	224816	.75139 9450	236249	.77629 4245	239198
3.1	.70399 9546	198937	.73672 9333	215111	.76290 0355	221609
3.2	.68759 1096	173065	.72184 4119	193897	.74928 4878	203904
3.3	.67100 9569	147265	.70676 5013	172659	.73546 5511	186124
3.4	.65428 0754	121594	.69151 3244	151439	.72146 0027	168300
3.5	0.63743 0309	−96113	0.67611 0025	−130280	0.70728 6246	−150466
3.6	.62048 3705	70882	.66057 6506	109232	.69296 1994	132660
3.7	.60346 6163	45955	.64493 3727	88333	.67850 5071	114913
3.8	.58640 2599	−21389	.62920 2580	67628	.66393 3218	97260
3.9	.56931 7571	+2763	.61340 3762	47159	.64926 4081	79734
4.0	.55223 5221	26449	.59755 7734	26966	.63451 5182	62367
4.1	.53517 9225	49622	.58168 4682	−7088	.61970 3881	45189
4.2	.51817 2748	72232	.56580 4477	+12438	.60484 7350	28232
4.3	.50123 8392	94236	.54993 6640	31575	.58996 2541	−11525
4.4	.48439 8153	115591	.53410 0299	50289	.57506 6155	+4903
4.5	.46767 3377	136256	.51831 4163	68544	.56017 4616	21024

* These differences are modified.

$$f_n(x) = \frac{n!}{(x/2)^n} J_n(x)$$

x	f_6	$*\delta^2$	f_7	$*\delta^2$	f_8	$*\delta^2$
4.5	0.46767 3377	136256	0.51831 4163	68544	0.56017 4616	21024
4.6	.45108 4725	156197	.50259 6481	86313	.54530 4041	36814
4.7	.43465 2129	175376	.48696 5017	103563	.53047 0213	52246
4.8	.41839 4763	193764	.47143 7016	120268	.51568 8562	67296
4.9	.40233 1012	211332	.45602 9176	136401	.50097 4132	81942
5.0	.38647 8436	228052	.44075 7628	151938	.48634 1565	96161
5.1	.37085 3753	243904	.42563 7905	166858	.47180 5078	109936
5.2	.35547 2810	258865	.41068 4922	181139	.45737 8441	123244
5.3	.34035 0566	272923	.39591 2957	194767	.44307 4961	136075
5.4	.32550 1075	286056	.38133 5636	207719	.42890 7463	148403
5.5	0.31093 7469	298260	0.36696 5908	219989	0.41488 8274	160223
5.6	.29667 1951	309525	.35281 6040	231560	.40102 9211	171517
5.7	.28271 5783	319843	.33889 7602	242423	.38734 1566	182274
5.8	.26907 9285	329216	.32522 1456	252572	.37383 6095	192486
5.9	.25577 1829	337641	.31179 7748	261999	.36052 3008	202143
6.0	.24280 1840	345123	.29863 5906	270701	.34741 1961	211239
6.1	.23017 6803	351669	.28574 4631	278678	.33451 2049	219769
6.2	.21790 3264	357284	.27313 1900	285926	.32183 1801	227726
6.3	.20598 6840	361983	.26080 4962	292453	.30937 9173	235111
6.4	.19443 2232	365778	.24877 0345	298258	.29716 1551	241921
6.5	0.18324 3239	368684	0.23703 3854	303349	0.28518 5744	248156
6.6	.17242 2770	370720	.22560 0582	307733	.27345 7988	253819
6.7	.16197 2864	371907	.21447 4915	311419	.26198 3946	258911
6.8	.15189 4714	372267	.20366 0541	314419	.25076 8711	263439
6.9	.14218 8684	371824	.19316 0462	316744	.23981 6811	267404
7.0	.13285 4334	370604	.18297 7006	318409	.22913 2214	270818
7.1	.12389 0451	368635	.17311 1840	319430	.21871 8334	273686
7.2	.11529 5071	365945	.16356 5989	319820	.20857 8041	276016
7.3	.10706 5509	362567	.15433 9846	319602	.19871 3667	277820
7.4	.09919 8394	358530	.14543 3195	318791	.18912 7018	279109
7.5	0.09168 9693	353868	0.13684 5231	317410	0.17981 9386	279895
7.6	.08453 4752	348615	.12857 4575	315478	.17079 1559	280190
7.7	.07772 8324	342805	.12061 9301	313018	.16204 3833	280008
7.8	.07126 4605	336475	.11297 6952	310054	.15357 6031	279365
7.9	.06513 7272	329658	.10564 4569	306608	.14538 7512	278275
8.0	.05933 9514	322391	.09861 8711	302705	.13747 7189	276755
8.1	.05386 4071	314713	.09189 5477	298370	.12984 3545	274821
8.2	.04870 3271	306656	.08547 0538	293626	.12248 4649	272490
8.3	.04384 9066	298261	.07933 9156	288503	.11539 8174	269782
8.4	.03929 3065	289561	.07349 6210	283021	.10858 1414	266711
8.5	0.03502 6576	280594	0.06793 6226	277212	0.10203 1302	263299
8.6	.03104 0638	271395	.06265 3397	271098	.09574 4429	259563
8.7	.02732 6059	261999	.05764 1615	264707	.08971 7063	255523
8.8	.02387 3450	252442	.05289 4494	258064	.08394 5169	251198
8.9	.02067 3258	242756	.04840 5395	251195	.07842 4423	246607
9.0	.01771 5805	232975	.04416 7454	244127	.07315 0239	241770
9.1	.01499 1315	223130	.04017 3607	236883	.06811 7782	236705
9.2	.01248 9949	213255	.03641 6617	229491	.06332 1993	231433
9.3	.01020 1839	203378	.03288 9093	221972	.05875 7602	225973
9.4	.00811 7110	193518	.02958 3522	214352	.05441 9152	220343
9.5	0.00622 5920	183732	0.02649 2290	206654	0.05030 1017	214562
9.6	.00451 8477	174019	.02360 7701	198902	.04639 7421	208651
9.7	.00298 5072	164412	.02092 2009	191117	.04270 2454	202625
9.8	.00161 6103	154935	.01842 7431	183320	.03921 0095	196504
9.9	+0.00040 2098	145612	.01611 6175	175531	.03591 4225	190304
10.0	−0.00066 6264	136462	.01398 0455	167770	.03280 8649	184046

* These differences are modified.

$$f_n(x) = \frac{n!}{(x/2)^n} J_n(x)$$

x	f_9	$*\delta^2$	f_{10}	$*\delta^2$	f_{11}	$*\delta^2$
0.0	1.00000 0000	−500069	1.00000 0000	−454603	1.00000 0000	−416715
0.1	0.99975 0028	499727	0.99977 2751	454318	0.99979 1687	416474
0.2	.99900 0455	498706	.99909 1288	453467	.99916 6987	415754
0.3	.99775 2300	497004	.99795 6462	452048	.99812 6622	414553
0.4	.99600 7265	494628	.99636 9691	450068	.99667 1790	412878
0.5	0.99376 7725	−491581	0.99433 2955	−447527	0.99480 4168	−410727
0.6	.99103 6726	487869	.99184 8793	444432	.99252 5906	408105
0.7	.98781 7979	483503	.98892 0300	440789	.98983 9624	405020
0.8	.98411 5848	478487	.98555 1119	436602	.98674 8406	401473
0.9	.97993 5348	472835	.98174 5432	431884	.98325 5799	397474
1.0	0.97528 2127	−466557	0.97750 7959	−426639	0.97936 5800	−393027
1.1	.97016 2463	459668	.97284 3941	420882	.97508 2854	388143
1.2	.96458 3241	452179	.96775 9134	414618	.97041 1846	382827
1.3	.95855 1946	444108	.96225 9799	407865	.96535 8087	377091
1.4	.95207 6648	435471	.95635 2688	400631	.95992 7313	370945
1.5	0.94516 5979	−426285	0.95004 5031	−392932	0.95412 5667	−364399
1.6	.93782 9123	416570	.94334 4525	384783	.94795 9694	357465
1.7	.93007 5791	406343	.93625 9316	376197	.94143 6326	350153
1.8	.92191 6205	395629	.92879 7986	367193	.93456 2871	342480
1.9	.91336 1077	384445	.92096 9538	357785	.92734 7001	334455
2.0	0.90442 1586	−372817	0.91278 3374	−347993	0.91979 6739	−326094
2.1	.89510 9356	360765	.90424 9286	337832	.91192 0441	317411
2.2	.88543 6434	348316	.89537 7429	327324	.90372 6788	308422
2.3	.87541 5265	335494	.88617 8308	316487	.89522 4768	299140
2.4	.86505 8666	322323	.87666 2757	305340	.88642 3658	289583
2.5	0.85437 9803	−308829	0.86684 1919	−293904	0.87733 3012	−279766
2.6	.84339 2165	295040	.85672 7227	282199	.86796 2647	269705
2.7	.83210 9536	280981	.84633 0382	270247	.85832 2617	259419
2.8	.82054 5971	266680	.83566 3330	258069	.84842 3208	248922
2.9	.80871 5766	252162	.82473 8248	245685	.83827 4912	238233
3.0	0.79663 3433	−237460	0.81356 7514	−233121	0.82788 8414	−227373
3.1	.78431 3669	222595	.80216 3689	220394	.81727 4572	216353
3.2	.77177 1335	207598	.79053 9496	207528	.80644 4404	205195
3.3	.75902 1421	192498	.77870 7796	194546	.79540 9063	193916
3.4	.74607 9024	177321	.76668 1567	181471	.78417 9825	182535
3.5	0.73295 9316	−162093	0.75447 3881	−168321	0.77276 8067	−171068
3.6	.71967 7518	146845	.74209 7883	155123	.76118 5253	159536
3.7	.70624 8874	131601	.72956 6768	141895	.74944 2912	147954
3.8	.69268 8623	116390	.71689 3759	128661	.73755 2623	136341
3.9	.67901 1972	101236	.70409 2087	115441	.72552 5996	124715
4.0	0.66523 4069	−86168	0.69117 4966	−102259	0.71337 4652	−113094
4.1	.65136 9979	71208	.67815 5575	89132	.70111 0211	101493
4.2	.63743 4655	56384	.66504 7039	76083	.68874 4269	89932
4.3	.62344 2918	41718	.65186 2400	63132	.67628 8385	78427
4.4	.60940 9430	27235	.63861 4608	50298	.66375 4061	66993
4.5	0.59534 8669	−12958	0.62531 6493	−37601	0.65115 2729	−55648
4.6	.58127 4909	+1093	.61198 0749	25058	.63849 5729	44407
4.7	.56720 2196	14894	.59861 9915	12689	.62579 4301	33285
4.8	.55314 4329	28426	.58524 6357	−511	.61305 9563	22298
4.9	.53911 4834	41669	.57187 2250	+11460	.60030 2499	11461
5.0	0.52512 6951	54604	0.55850 9562	23207	0.58753 3945	−787

* These differences are modified.

$$f_n(x) = \frac{n!}{(x/2)^n} J_n(x)$$

x	f_9	$*\delta^2$	f_{10}	$*\delta^2$	f_{11}	$*\delta^2$
5.0	0.52512 6951	54604	0.55850 9562	23207	0.58753 3945	− 787
5.1	.51119 3613	67214	.54517 0037	34714	.57476 4571	+ 9711
5.2	.49732 7426	79480	.53186 5180	45966	.56200 4873	20017
5.3	.48354 0653	91391	.51860 6240	56953	.54926 5156	30123
5.4	.46984 5203	102926	.50540 4201	67653	.53655 5521	40012
5.5	0.45625 2608	114077	0.49226 9760	78062	0.52388 5857	49676
5.6	.44277 4016	124828	.47921 3326	88164	.51126 5825	59103
5.7	.42942 0177	135168	.46624 4996	97947	.49870 4851	68282
5.8	.41620 1428	145088	.45337 4554	107404	.48621 2113	77205
5.9	.40312 7687	154576	.44061 1454	116524	.47379 6530	85863
6.0	0.39020 8442	163626	0.42796 4813	125297	0.46146 6760	94245
6.1	.37745 2741	172231	.41544 3405	133718	.44923 1183	102347
6.2	.36486 9187	180382	.40305 5649	141778	.43709 7900	110158
6.3	.35246 5932	188079	.39080 9604	149473	.42507 4721	117676
6.4	.34025 0670	195313	.37871 2963	156796	.41316 9163	124891
6.5	0.32823 0637	202084	0.36677 3049	163743	0.40138 8439	131800
6.6	.31641 2602	208391	.35499 6808	170311	.38973 9458	138400
6.7	.30480 2872	214230	.34339 0808	176496	.37822 8819	144683
6.8	.29340 7287	219606	.33196 1233	182299	.36686 2806	150651
6.9	.28223 1222	224516	.32071 3886	187716	.35564 7384	156297
7.0	0.27127 9589	228965	0.30965 4184	192748	0.34458 8201	161622
7.1	.26055 6836	232956	.29878 7159	197395	.33369 0580	166625
7.2	.25006 6955	236491	.28811 7459	201659	.32295 9525	171302
7.3	.23981 3483	239579	.27764 9348	205542	.31239 9712	175657
7.4	.22979 9508	242222	.26738 6709	209046	.30201 5497	179689
7.5	0.22002 7675	244431	0.25733 3047	212175	0.29181 0912	183399
7.6	.21050 0194	246209	.24749 1492	214932	.28178 9667	186788
7.7	.20121 8844	247568	.23786 4801	217324	.27195 5151	189861
7.8	.19218 4988	248517	.22845 5368	219355	.26231 0438	192618
7.9	.18339 9575	249063	.21926 5224	221030	.25285 8286	195064
8.0	0.17486 3153	249219	0.21029 6046	222357	0.24360 1142	197203
8.1	.16657 5880	248995	.20154 9163	223344	.23454 1146	199039
8.2	.15853 7534	248403	.19302 5562	223996	.22568 0133	200577
8.3	.15074 7527	247456	.18472 5898	224325	.21701 9644	201823
8.4	.14320 4912	246163	.17665 0500	224334	.20856 0924	202779
8.5	0.13590 8399	244542	0.16879 9380	224038	0.20030 4933	203457
8.6	.12885 6371	242603	.16117 2243	223443	.19225 2348	203860
8.7	.12204 6890	240361	.15376 8495	222559	.18440 3573	203995
8.8	.11547 7716	237829	.14658 7255	221397	.17675 8747	203871
8.9	.10914 6321	235023	.13962 7363	219967	.16931 7744	203493
9.0	0.10304 9902	231956	0.13288 7391	218280	0.16208 0189	202871
9.1	.09718 5393	228643	.12636 5653	216346	.15504 5462	202012
9.2	.09154 9484	225099	.12006 0219	214178	.14821 2705	200924
9.3	.08613 8635	221339	.11396 8922	211785	.14158 0831	199616
9.4	.08094 9087	217377	.10808 9370	209181	.13514 8535	198098
9.5	0.07597 6882	213229	0.10241 8963	206375	0.12891 4299	196377
9.6	.07121 7874	208909	.09695 4896	203380	.12287 6405	194462
9.7	.06666 7746	204431	.09169 4176	200207	.11703 2939	192364
9.8	.06232 2023	199812	.08663 3633	196869	.11138 1804	190090
9.9	.05817 6089	195065	.08176 9931	193377	.10592 0729	187651
10.0	0.05422 5199	190205	0.07709 9580	189741	0.10064 7277	* 185056

* These differences are modified.

$$f_n(x) = \frac{n!}{(x/2)^n} J_n(x)$$

x	f_{12}	$*\delta^2$	f_{13}	$*\delta^2$	f_{14}	$*\delta^2$
0.0	1.00000 0000	−384657	1.00000 0000	−357179	1.00000 0000	−333365
0.1	0.99980 7710	384451	0.99982 1444	357000	0.99983 3346	333209
0.2	.99923 1044	383834	.99928 5952	356465	.99933 3542	332740
0.3	.99827 0621	382804	.99839 4062	355573	.99850 1054	331959
0.4	.99692 7469	381367	.99714 6664	354327	.99733 6664	330870
0.5	0.99520 3024	−379523	0.99554 5003	−352728	0.99584 1461	−329470
0.6	.99309 9131	377274	.99359 0678	350779	.99401 6845	327763
0.7	.99061 8037	374628	.99128 5638	348484	.99186 4522	325754
0.8	.98776 2388	371584	.98863 2178	345843	.98938 6500	323441
0.9	.98453 5226	368152	.98563 2938	342864	.98658 5092	320832
1.0	0.98093 9984	−364332	0.98229 0894	−339549	0.98346 2906	−317927
1.1	.97698 0478	360137	.97860 9362	335907	.98002 2846	314734
1.2	.97266 0904	355569	.97459 1984	331938	.97626 8104	311255
1.3	.96798 5828	350637	.97024 2726	327652	.97220 2159	307496
1.4	.96296 0180	345349	.96556 5873	323054	.96782 8768	303462
1.5	0.95758 9246	−339715	0.96056 6021	−318153	0.96315 1966	−299159
1.6	.95187 8661	333742	.95524 8071	312955	.95817 6053	294594
1.7	.94583 4393	327441	.94961 7219	307467	.95290 5593	289771
1.8	.93946 2743	320823	.94367 8950	301701	.94734 5408	284701
1.9	.93277 0327	313896	.93743 9031	295661	.94150 0566	279388
2.0	0.92576 4069	−306674	0.93090 3499	−289360	0.93537 6379	−273842
2.1	.91845 1189	299167	.92407 8654	282805	.92897 8391	268068
2.2	.91083 9192	291389	.91697 1049	276007	.92231 2375	262077
2.3	.90293 5854	283350	.90958 7479	268977	.91538 4320	255876
2.4	.89474 9211	275064	.90193 4973	261724	.90820 0427	249474
2.5	0.88628 7548	−266544	0.89402 0782	−254260	0.90076 7094	−242880
2.6	.87755 9381	257803	.88585 2368	246594	.89309 0915	236103
2.7	.86857 3448	248856	.87743 7394	238741	.88517 8665	229154
2.8	.85933 8696	239716	.86878 3713	230709	.87703 7291	222040
2.9	.84986 4260	230395	.85989 9353	222510	.86867 3905	214771
3.0	0.84015 9460	−220913	0.85079 2512	−214159	0.86009 5775	−207361
3.1	.83023 3774	211279	.84147 1537	205665	.85131 0308	199815
3.2	.82009 6833	201510	.83194 4922	197041	.84232 5048	192146
3.3	.80975 8406	191621	.82222 1287	188299	.83314 7662	184364
3.4	.79922 8376	181626	.81230 9373	179453	.82378 5932	176480
3.5	0.78851 6738	−171540	0.80221 8022	−170514	0.81424 7739	−168502
3.6	.77763 3573	161379	.79195 6173	161494	.80454 1058	160443
3.7	.76658 9041	151156	.78153 2841	152407	.79467 3948	152313
3.8	.75539 3362	140887	.77095 7114	143264	.78465 4536	144121
3.9	.74405 6800	130587	.76023 8130	134078	.77449 1012	135881
4.0	0.73258 9654	−120272	0.74938 5074	−124863	0.76419 1614	−127600
4.1	.72100 2237	109952	.73840 7159	115628	.75376 4621	119291
4.2	.70930 4866	99646	.72731 3618	106387	.74321 8341	110962
4.3	.69750 7844	89366	.71611 3688	97153	.73256 1101	102626
4.4	.68562 1448	79126	.70481 6602	87937	.72180 1234	94292
4.5	0.67365 5917	− 68941	0.69343 1574	−78751	0.71094 7073	− 85969
4.6	.66162 1432	58824	.68196 7787	69606	.70000 6939	77670
4.7	.64952 8108	48787	.67043 4385	60514	.68898 9128	69402
4.8	.63738 5980	38845	.65884 0458	51486	.67790 1908	61176
4.9	.62520 4988	29009	.64719 5031	42533	.66675 3503	53002
5.0	0.61299 4965	− 19292	0.63550 7055	−33666	0.65555 2084	− 44889

* These differences are modified.

$$f_n(x) = \frac{n!}{(x/2)^n} J_n(x)$$

x	f_{12}	$*\delta^2$	f_{13}	$*\delta^2$	f_{14}	$*\delta^2$
5.0	0.61299 4965	−19292	0.63550 7055	−33666	0.65555 2084	−44889
5.1	.60076 5626	9705	.62378 5396	24895	.64430 5764	36845
5.2	.58852 6556	− 262	.61203 8823	16231	.63302 2584	28881
5.3	.57628 7195	+ 9030	.60027 5998	− 7681	.62171 0507	21003
5.4	.56405 6834	18155	.58850 5468	+ 740	.61037 7410	13223
5.5	0.55184 4597	27108	0.57673 5654	9029	0.59903 1070	− 5547
5.6	.53965 9433	35876	.56497 4843	17173	.58767 9163	+ 2018
5.7	.52751 0109	44450	.55323 1177	25165	.57632 9252	9461
5.8	.51540 5198	52822	.54151 2647	32996	.56498 8778	16778
5.9	.50335 3070	60983	.52982 7082	40660	.55366 5058	23961
6.0	0.49136 1884	68924	0.51818 2144	48147	0.54236 5273	31001
6.1	.47943 9580	76641	.50658 5321	55453	.53109 6461	37894
6.2	.46759 3875	84123	.49504 3917	62569	.51986 5515	44633
6.3	.45583 2248	91366	.48356 5045	69491	.50867 9172	51212
6.4	.44416 1942	98364	.47215 5627	76210	.49754 4012	57626
6.5	0.43258 9954	105111	0.46082 2382	82724	0.48646 6446	63869
6.6	.42112 3030	111602	.44957 1821	89026	.47545 2716	69937
6.7	.40976 7659	117833	.43841 0247	95112	.46450 8891	75825
6.8	.39853 0073	123801	.42734 3745	100979	.45364 0856	81529
6.9	.38741 6239	129501	.41637 8181	106622	.44285 4316	87044
7.0	0.37643 1856	134932	0.40551 9197	112038	0.43215 4785	92369
7.1	.36558 2354	140091	.39477 2209	117225	.42154 7588	97500
7.2	.35487 2894	144975	.38414 2402	122178	.41103 7854	102432
7.3	.34430 8358	149586	.37363 4732	126899	.40063 0515	107165
7.4	.33389 3357	153920	.36325 3917	131384	.39033 0304	111697
7.5	0.32363 2226	157979	0.35300 4442	135632	0.38014 1752	116024
7.6	.31352 9023	161762	.34289 0556	139643	.37006 9188	120147
7.7	.30358 7532	165270	.33291 6269	143415	.36011 6732	124065
7.8	.29381 1260	168505	.32308 5354	146950	.35028 8304	127775
7.9	.28420 3443	171467	.31340 1346	150248	.34058 7612	131278
8.0	0.27476 7044	174160	0.30386 7542	153309	0.33101 8160	134574
8.1	.26550 4755	176585	.29448 7003	156135	.32158 3243	137663
8.2	.25641 9003	178745	.28526 2557	158726	.31228 5952	140546
8.3	.24751 1947	180646	.27619 6794	161087	.30312 9168	143223
8.4	.23878 5490	182286	.26729 2076	163216	.29411 5570	145695
8.5	0.23024 1272	183675	0.25855 0532	165119	0.28524 7629	147965
8.6	.22188 0684	184815	.24997 4067	166798	.27652 7617	150034
8.7	.21370 4865	185710	.24156 4358	168255	.26795 7602	151903
8.8	.20571 4712	186366	.23332 2865	169495	.25953 9453	153575
8.9	.19791 0882	186787	.22525 0827	170520	.25127 4844	155052
9.0	0.19029 3797	186981	0.21734 9271	171336	0.24316 5252	156339
9.1	.18286 3652	186953	.20961 9013	171946	.23521 1964	157436
9.2	.17562 0420	186707	.20206 0663	172354	.22741 6078	158348
9.3	.16856 3858	186253	.19467 4631	172566	.21977 8507	159077
9.4	.16169 3511	185596	.18746 1130	172586	.21229 9979	159628
9.5	0.15500 8724	184741	0.18042 0179	172419	0.20498 1048	160005
9.6	.14850 8643	183698	.17355 1615	172072	.19782 2090	160210
9.7	.14219 2226	182471	.16685 5091	171548	.19082 3312	160249
9.8	.13605 8248	181070	.16033 0083	170855	.18398 4753	160126
9.9	.13010 5309	179501	.15397 5899	169997	.17730 6290	159845
10.0	0.12433 1842	177771	0.14779 1683	168979	0.17078 7645	159411

* These differences are modified.

$$f_n(x) = \frac{n!}{(x/2)^n} J_n(x)$$

x	f_{15}	$*\delta^2$	f_{16}	$*\delta^2$	f_{17}	$*\delta^2$
0.0	1.00000 0000	−312528	1.00000 0000	−294142	1.00000 0000	−277800
0.1	0.99984 3762	312390	0.99985 2951	294020	0.99986 1120	277690
0.2	.99937 5184	311977	.99941 1928	293653	.99944 4591	277362
0.3	.99859 4680	311287	.99867 7297	293040	.99875 0740	276813
0.4	.99750 2939	310326	.99764 9671	292185	.99778 0115	276048
0.5	0.99610 0922	−309091	0.99632 9905	−291087	0.99653 3483	−275065
0.6	.99438 9865	307584	.99471 9097	289747	.99501 1824	273867
0.7	.99237 1273	305810	.99281 8585	288169	.99321 6339	272454
0.8	.99004 6920	303768	.99062 9948	286353	.99114 8439	270828
0.9	.98741 8847	301463	.98815 5002	284302	.98880 9749	268992
1.0	0.98448 9360	−298897	0.98539 5795	−282019	0.98620 2106	−266947
1.1	.98126 1022	296076	.98235 4612	279508	.98332 7554	264697
1.2	.97773 6655	293000	.97903 3963	276769	.98018 8342	262244
1.3	.97391 9333	289676	.97543 6585	273809	.97678 6923	259590
1.4	.96981 2381	286108	.97156 5439	270630	.97312 5951	256740
1.5	0.96541 9365	−282300	0.96742 3702	−267237	0.96920 8274	−253697
1.6	.96074 4092	278259	.96301 4767	263634	.96503 6936	250465
1.7	.95579 0603	273987	.95834 2237	259824	.96061 5167	247046
1.8	.95056 3167	269495	.95340 9919	255816	.95594 6385	243448
1.9	.94506 6275	264785	.94822 1821	251611	.95103 4189	239670
2.0	0.93930 4637	−259865	0.94278 2146	−247217	0.94588 2354	−235723
2.1	.93328 3172	254741	.93709 5288	242639	.94049 4827	231606
2.2	.92700 7003	249420	.93116 5825	237882	.93487 5723	227328
2.3	.92048 1448	243910	.92499 8511	232953	.92902 9320	222893
2.4	.91371 2016	238217	.91859 8275	227857	.92296 0052	218305
2.5	0.90670 4400	−232349	0.91197 0212	−222602	0.91667 2506	−213571
2.6	.89946 4465	226314	.90511 9575	217194	.91017 1414	208697
2.7	.89199 8245	220121	.89805 1770	211640	.90346 1650	203687
2.8	.88431 1932	213777	.89077 2353	205945	.89654 8223	198549
2.9	.87641 1868	207289	.88328 7014	200118	.88943 6270	193287
3.0	0.86830 4539	−200668	0.87560 1579	−194168	0.88213 1051	−187909
3.1	.85999 6566	193921	.86772 1999	188098	.87463 7945	182420
3.2	.85149 4694	187056	.85965 4341	181918	.86696 2437	176827
3.3	.84280 5785	180084	.85140 4783	175635	.85911 0120	171136
3.4	.83393 6811	173013	.84297 9607	169258	.85108 6684	165355
3.5	0.82489 4841	−165850	0.83438 5190	−162792	0.84289 7908	−159489
3.6	.81568 7036	158606	.82562 7995	156246	.83454 9657	153546
3.7	.80632 0638	151291	.81671 4568	149629	.82604 7873	147531
3.8	.79680 2961	143911	.80765 1523	142946	.81739 8570	141452
3.9	.78714 1383	136477	.79844 5542	136208	.80860 7825	135317
4.0	0.77734 3337	−128998	0.78910 3363	−129421	0.79968 1772	−129131
4.1	.76741 6298	121483	.77963 1770	122592	.79062 6598	122899
4.2	.75736 7782	113940	.77003 7591	115731	.78144 8531	116632
4.3	.74720 5329	106380	.76032 7686	108845	.77215 3837	110335
4.4	.73693 6497	98810	.75050 8939	101940	.76274 8813	104014
4.5	0.72656 8856	− 91239	0.74058 8255	− 95026	0.75323 9779	− 97676
4.6	.71610 9975	83677	.73057 2545	88110	.74363 3069	91329
4.7	.70556 7414	76131	.72046 8724	81199	.73393 5032	84978
4.8	.69494 8717	68610	.71028 3701	74301	.72415 2016	78631
4.9	.68426 1404	61124	.70002 4374	67423	.71429 0367	72294
5.0	0.67351 2959	− 53679	0.68969 7620	− 60572	0.70435 6422	− 65972

* These differences are modified.

$$f_n(x) = \frac{n!}{(x/2)^n} J_n(x)$$

x	f_{15}	$*\delta^2$	f_{16}	$*\delta^2$	f_{17}	$*\delta^2$
5.0	0.67351 2959	−53679	0.68969 7620	−60572	0.70435 6422	−65972
5.1	.66271 0827	46284	.67931 0287	53756	.69435 6500	59674
5.2	.65186 2399	38947	.66886 9190	46982	.68429 6899	53404
5.3	.64097 5013	31675	.65838 1102	40255	.67418 3887	47169
5.4	.63005 5937	24478	.64785 2749	33586	.66402 3699	40976
5.5	0.61911 2369	−17360	0.63729 0798	−26977	0.65382 2525	−34830
5.6	.60815 1424	10330	.62670 1858	20436	.64358 6512	28736
5.7	.59718 0131	− 3396	.61609 2469	13971	.63332 1753	22701
5.8	.58620 5424	+ 3438	.60546 9095	7585	.62303 4280	16730
5.9	.57523 4136	10165	.59483 8119	− 1286	.61273 0065	10828
6.0	0.56427 2992	16776	0.58420 5841	+ 4920	0.60241 5009	− 5001
6.1	.55332 8601	23268	.57357 8466	11030	.59209 4938	+ 747
6.2	.54240 7456	29634	.56296 2101	17035	.58177 5598	6410
6.3	.53151 5920	35869	.55236 2751	22934	.57146 2652	11985
6.4	.52066 0228	41967	.54178 6314	28718	.56116 1674	17465
6.5	0.50984 6477	47924	0.53123 8574	34385	0.55087 8143	22847
6.6	.49908 0622	53734	.52072 5195	39930	.54061 7440	28128
6.7	.48836 8474	59394	.51025 1724	45348	.53038 4845	33301
6.8	.47771 5692	64900	.49982 3577	50638	.52018 5531	38365
6.9	.46712 7781	70247	.48944 6043	55792	.51002 4560	43315
7.0	0.45661 0087	75432	0.47912 4276	60809	0.49990 6884	48148
7.1	.44616 7795	80452	.46886 3291	65685	.48983 7332	52861
7.2	.43580 5924	85304	.45866 7965	70417	.47982 0620	57450
7.3	.42552 9325	89985	.44854 3030	75004	.46986 1334	61914
7.4	.41534 2679	94493	.43849 3070	79441	.45996 3939	66250
7.5	0.40525 0494	98825	0.42852 2524	83726	0.45013 2770	70453
7.6	.39525 7101	102980	.41863 5675	87858	.44037 2029	74524
7.7	.38536 6655	106957	.40883 6655	91835	.43068 5787	78460
7.8	.37558 3132	110754	.39912 9441	95655	.42107 7980	82259
7.9	.36591 0330	114369	.38951 7853	99316	.41155 2407	85919
8.0	0.35635 1863	117804	0.38000 5553	102820	0.40211 2726	89440
8.1	.34691 1167	121055	.37059 6043	106162	.39276 2459	92818
8.2	.33759 1492	124126	.36129 2664	109344	.38350 4985	96056
8.3	.32839 5910	127015	.35209 8600	112365	.37434 3541	99152
8.4	.31932 7310	129721	.34301 6872	115223	.36528 1222	102102
8.5	0.31038 8397	132247	0.33405 0337	117922	0.35632 0979	104910
8.6	.30158 1698	134593	.32520 1695	120460	.34746 5620	107575
8.7	.29290 9558	136759	.31647 3483	122836	.33871 7809	110095
8.8	.28437 4145	138749	.30786 8079	125054	.33008 0067	112471
8.9	.27597 7447	140561	.29938 7699	127111	.32155 4769	114704
9.0	0.26772 1279	142200	0.29103 4401	129012	0.31314 4149	116795
9.1	.25960 7280	143667	.28281 0086	130756	.30485 0299	118744
9.2	.25163 6916	144963	.27471 6499	132344	.29667 5166	120551
9.3	.24381 1485	146093	.26675 5228	133780	.28862 0559	122219
9.4	.23613 2117	147057	.25892 7710	135064	.28068 8145	123748
9.5	0.22859 9777	147859	0.25123 5228	136199	0.27287 9454	125139
9.6	.22121 5267	148502	.24367 8917	137185	.26519 5877	126395
9.7	.21397 9230	148990	.23625 9765	138028	.25763 8670	127516
9.8	.20689 2155	149324	.22897 8615	138726	.25020 8955	128505
9.9	.19995 4376	149509	.22183 6165	139286	.24290 7721	129364
10.0	0.19316 6080	149550	0.21483 2976	139710	0.23573 5828	130094

* These differences are modified.

$$f_n(x) = \frac{n!}{(x/2)^n} J_n(x)$$

x	f_{18}	$*\delta^2$	f_{19}	$*\delta^2$	f_{20}	$*\delta^2$
0.0	1.00000 0000	−263178	1.00000 0000	−250018	1.00000 0000	−238111
0.1	0.99986 8429	263079	0.99987 5007	249929	0.99988 0959	238031
0.2	.99947 3816	262783	.99950 0119	249661	.99952 3918	237787
0.3	.99881 6455	262290	.99887 5603	249215	.99892 9119	237381
0.4	.99789 6841	261601	.99800 1904	248591	.99809 6969	236815
0.5	0.99671 5661	−260716	0.99687 9646	−247791	0.99702 8033	−236087
0.6	.99527 3801	259637	.99550 9630	246814	.99572 3041	235198
0.7	.99357 2338	258365	.99389 2832	245663	.99418 2878	234151
0.8	.99161 2546	256901	.99203 0402	244337	.99240 8594	232946
0.9	.98939 5888	255247	.98992 3668	242840	.99040 1392	231585
1.0	0.98692 4017	−253405	0.98757 4123	−241173	0.98816 2634	−230067
1.1	.98419 8774	251378	.98498 3438	239336	.98569 3838	228396
1.2	.98122 2188	249167	.98215 3447	237334	.98299 6674	226573
1.3	.97799 6467	246775	.97908 6153	235166	.98007 2963	224601
1.4	.97452 4004	244206	.97578 3723	232838	.97692 4679	222480
1.5	0.97080 7367	−241461	0.97224 8483	−230350	0.97355 3941	−220215
1.6	.96684 9302	238545	.96848 2923	227705	.96996 3015	217806
1.7	.96265 2722	235459	.96448 9686	224908	.96615 4309	215257
1.8	.95822 0713	232211	.96027 1569	221959	.96213 0371	212570
1.9	.95355 6522	228800	.95583 1520	218864	.95789 3888	209748
2.0	0.94866 3560	−225233	0.95117 2633	−215626	0.95344 7681	−206794
2.1	.94354 5393	221513	.94629 8146	212247	.94879 4703	203712
2.2	.93820 5740	217645	.94121 1438	208732	.94393 8036	200504
2.3	.93264 8469	213632	.93591 6022	205085	.93888 0888	197174
2.4	.92687 7591	209480	.93041 5545	201309	.93362 6587	193726
2.5	0.92089 7258	−205194	0.92471 3782	−197409	0.92817 8582	−190162
2.6	.91471 1755	200777	.91881 4632	193389	.92254 0434	186488
2.7	.90832 5498	196236	.91272 2116	189253	.91671 5819	182705
2.8	.90174 3027	191575	.90644 0366	185007	.91070 8517	178820
2.9	.89496 9002	186799	.89997 3630	180653	.90452 2414	174835
3.0	0.88800 8198	−181916	0.89332 6259	−176198	0.89816 1492	−170754
3.1	.88086 5496	176928	.88650 2709	171645	.89162 9834	166583
3.2	.87354 5885	171842	.87950 7530	167002	.88493 1608	162325
3.3	.86605 4448	166664	.87234 5366	162268	.87807 1072	157983
3.4	.85839 6364	161400	.86502 0948	157455	.87105 2567	153565
3.5	0.85057 6895	−156054	0.85753 9090	−152563	0.86388 0512	−149071
3.6	.84260 1386	150633	.84990 4682	147600	.85655 9397	144509
3.7	.83447 5256	145143	.84212 2686	142568	.84909 3786	139883
3.8	.82620 3995	139590	.83419 8134	137476	.84148 8302	135195
3.9	.81779 3154	133980	.82613 6116	132327	.83374 7634	130454
4.0	0.80924 8343	−128318	0.81794 1780	−127126	0.82587 6521	−125660
4.1	.80057 5222	122611	.80962 0326	121880	.81787 9757	120821
4.2	.79177 9497	116864	.80117 7001	116592	.80976 2179	115940
4.3	.78286 6915	111085	.79261 7089	111269	.80152 8668	111022
4.4	.77384 3252	105277	.78394 5914	105916	.79318 4141	106073
4.5	0.76471 4317	−99449	0.77516 8827	−100538	0.78473 3546	−101096
4.6	.75548 5936	93604	.76629 1205	95141	.77618 1859	96096
4.7	.74616 3954	87750	.75731 8445	89728	.76753 4079	91079
4.8	.73675 4222	81892	.74825 5960	84306	.75879 5222	86047
4.9	.72726 2597	76036	.73910 9168	78882	.74997 0320	81008
5.0	0.71769 4936	−70188	0.72988 3495	−73456	0.74106 4410	−75964

* These differences are modified.

$$f_n(x) = \frac{n!}{(x/2)^n} J_n(x)$$

x	f_{18}	$*\delta^2$	f_{19}	$*\delta^2$	f_{20}	$*\delta^2$
5.0	0.71769 4936	−70188	0.72988 3495	−73456	0.74106 4410	−75964
5.1	.70805 7084	64353	.72058 4365	68038	.73208 2537	70920
5.2	.69835 4876	58537	.71121 7195	62630	.72302 9743	65882
5.3	.68859 4127	52745	.70178 7392	57239	.71391 1065	60852
5.4	.67878 0627	46984	.69230 0345	51868	.70473 1533	55837
5.5	0.66892 0137	−41257	0.68276 1426	−46524	0.69549 6160	−50839
5.6	.65901 8383	35571	.67317 5977	41209	.68620 9945	45864
5.7	.64908 1049	29931	.66354 9313	35930	.67687 7861	40915
5.8	.63911 3775	24341	.65388 6712	30690	.66750 4856	35998
5.9	.62912 2151	18806	.64419 3412	25495	.65809 5847	31115
6.0	0.61911 1709	−13332	0.63447 4608	−20347	0.64865 5717	−26269
6.1	.60908 7924	7922	.62473 5447	15253	.63918 9309	21468
6.2	.59905 6205	− 2582	.61498 1023	10216	.62970 1425	16713
6.3	.58902 1891	+ 2686	.60521 6372	5238	.62019 6819	12007
6.4	.57899 0247	7874	.59544 6471	− 326	.61068 0195	7356
6.5	0.56896 6462	12981	0.58567 6232	+ 4518	0.60115 6205	− 2762
6.6	.55895 5643	18002	.57591 0497	9290	.59162 9442	+ 1772
6.7	.54896 2809	22934	.56615 4038	13986	.58210 4439	6241
6.8	.53899 2892	27772	.55641 1551	18603	.57258 5666	10645
6.9	.52905 0729	32514	.54668 7651	23138	.56307 7524	14977
7.0	0.51914 1062	37156	0.53698 6874	27588	0.55358 4346	19236
7.1	.50926 8533	41695	.52731 3667	31949	.54411 0391	23420
7.2	.49943 7679	46128	.51767 2393	36219	.53465 9842	27525
7.3	.48965 2933	50453	.50806 7321	40395	.52523 6803	31550
7.4	.47991 8620	54667	.49850 2626	44476	.51584 5298	35490
7.5	0.47023 8953	58767	0.48898 2388	48457	0.50648 9267	39344
7.6	.46061 8031	62750	.47951 0589	52337	.49717 2564	43110
7.7	.45105 9838	66617	.47009 1107	56114	.48789 8955	46786
7.8	.44156 8239	70363	.46072 7721	59787	.47867 2115	50370
7.9	.43214 6981	73988	.45142 4101	63352	.46949 5627	53860
8.0	0.42279 9688	77490	0.44218 3814	66810	0.46037 2982	57254
8.1	.41352 9862	80867	.43301 0316	70157	.45130 7574	60552
8.2	.40434 0880	84119	.42390 6954	73393	.44230 2699	63750
8.3	.39523 5993	87245	.41487 6965	76517	.43336 1556	66849
8.4	.38621 8328	90242	.40592 3472	79527	.42448 7243	69846
8.5	0.37729 0881	93112	0.39704 9486	82423	0.41568 2757	72742
8.6	.36845 6523	95854	.38825 7901	85205	.40695 0994	75535
8.7	.35971 7994	98466	.37955 1501	87870	.39829 4747	78223
8.8	.35107 7909	100949	.37093 2949	90421	.38971 6704	80808
8.9	.34253 8748	103304	.36240 4796	92852	.38121 9450	83288
9.0	0.33410 2868	105529	0.35396 9475	95171	0.37280 5465	85662
9.1	.32577 2494	107626	.34562 9303	97371	.36447 7122	87932
9.2	.31754 9721	109595	.33738 6480	99455	.35623 6692	90094
9.3	.30943 6520	111435	.32924 3091	101422	.34808 6337	92153
9.4	.30143 4731	113150	.32120 1102	103275	.34002 8116	94105
9.5	0.29354 6068	114738	0.31326 2367	105011	0.33206 3980	95952
9.6	.28577 2121	116201	.30542 8623	106635	.32419 5777	97694
9.7	.27811 4352	117541	.29770 1492	108142	.31642 5248	99331
9.8	.27057 4103	118758	.29008 2482	109539	.30875 4032	100864
9.9	.26315 2589	119855	.28257 2991	110822	.30118 3661	102295
10.0	0.25585 0909	120832	0.27517 4301	111994	0.29371 5566	103625

* These differences are modified.

A BOUNDARY VALUE PROBLEM IN PLANE PLASTICITY*

By N. Coburn

1. Introduction. The following problem is discussed in this paper. Given a perfectly plastic material occupying the semi-space $x > 0$. The stresses $\sigma_x(0, y)$, $\sigma_{xy}(0, y)$ acting on the boundary $x = 0$ are considered as known. Our problem is to determine the stresses acting at each point of the semi-space $x > 0$.

The determination of the stresses is known to depend upon the solution of a nonlinear second order partial differential equation of the hyperbolic type in the x, y, plane [1][1]. The problem has been attacked from various viewpoints. In 1871, M. Levy [2] studied the mappings of the x, y physical plane into the plane of the variables σ, γ, where σ denotes the mean normal stress and γ is the angle between the tangent to one family of slip lines (characteristics of the above mentioned nonlinear differential equation) and the x-axis. He showed that the mapping functions could be expressed in terms of the hyperbolic and ordinary sine and cosine functions. Next, the geometric properties of the orthogonal net of slip lines, ξ = constant and η = constant, were investigated. In 1923, H. Hencky [3] and L. Prandtl [4] proved that the following two properties are valid for this net: (1) the two tangents to any curve of one family, which cut any two curves of the other family, are so oriented that the angle between the two tangents is constant; (2) at any point, the rate of change of the radius of curvature of a curve of one family with respect to the arc length along the curve of the other family, which passes through the given point, has the magnitude one. By use of these properties, C. Caratheodory and E. Schmidt [5] gave an analytic method for determining this net when one curve of each family is known. This contribution is equivalent to determining the mapping functions from the x, y plane to the ξ, η plane when the boundary conditions are given along two intersecting characteristics. Other interesting geometric properties of the net were given by R. v. Mises [6], W. Prager [7], and I. Capuano [8]. One of v. Mises' results is of considerable interest from our standpoint since it connects the work of Levy and later contributors, namely, the diagonals of the net σ = constant and γ = constant form the net ξ = constant and η = constant. Recently, H. Geiringer and W. Prager [9], S. Christianovich [10], and S. Bergman [11] have made important contributions by an analytical approach to the problem. Geiringer and Prager showed that the functions mapping the x, y plane into the ξ, η plane are the solutions of a linear hyperbolic second order partial differential equation in which the first derivatives are missing. In a penetrating analysis, Christian-

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[1] The numbers in bracketts refer to the bibliography.

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ovich discussed the solutions of the Geiringer-Prager equations. Further, he introduced the idea of a line of rupture (defined as a curve along which one of the partial derivatives of the stresses becomes infinite) and showed that if the closed boundary curve and the boundary stresses satisfy certain regularity conditions then: (1) in the interior problem, there will always exist a line of rupture; (2) in the exterior problem, there will never exist a line of rupture. Finally, Bergman indicated a method for generating particular solutions of the Geiringer-Prager equations.

Our procedure will be to start with the three well known equations of plane plasticity [12] involving the stresses σ_x, σ_y, σ_{xy}, the mean normal stress σ, and the variable γ denoting the angle between the tangent to one family of slip lines and the x-axis. Upon eliminating the mean normal stress σ and replacing σ_x, σ_y, σ_{xy}, by the proper partial derivatives of the Airy stress function, we obtain a system of two linear partial differential equations of the hyperbolic type. One of these equations can be immediately integrated and furnishes the stresses in terms of: (1) the boundary conditions; (2) the sine function of 2γ. The remaining equation of the above system of linear partial differential equations can be solved and furnishes the cosine function of 2γ in terms of the boundary data. In view of the fact that cosine of 2γ lies in the range $(-1, 1)$, we can formulate a mathematical condition for the determination of the boundary "curve beyond which the statically determinate plastic stress distribution cannot be extended."[2] By use of our formulas for the stresses σ_x, σ_y, σ_{xy}, we can easily show that along this boundary curve one of the partial derivatives of the stresses $(\partial\sigma_y/\partial x)$ becomes infinite. Thus, this boundary curve is a line of rupture in the sense of Christianovich.

2. **Formulation of the problem.** We consider the semi-space $x > 0$. The material occupying this region is assumed to be perfectly plastic. At any point of the closed region R occupied by this plastic material, the following stress components exist: normal stresses σ_x and σ_y acting parallel to the x and y-axis, respectively; a shear stress σ_{xy}. The equilibrium relations can be satisfied by assuming the existence of the Airy stress function $F(x, y)$ such that

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \qquad \sigma_{xy} = -\frac{\partial^2 F}{\partial x \partial y}. \tag{2.1}$$

Further, the following relations are well known[3]

$$\sigma_x = \sigma + k \sin 2\gamma, \tag{2.2}$$

$$\sigma_y = \sigma - k \sin 2\gamma, \tag{2.3}$$

$$\sigma_{xy} = -k \cos 2\gamma, \tag{2.4}$$

[2] This interpretation of the line of rupture is due to Professor W. Prager.

[3] See [12].

where k is a characteristic constant of the material (the yield stress in shearing), σ is the mean normal stress, and γ is the angle between the tangent to one family of slip lines and the x-axis. Upon subtracting (2.3) from (2.2), we find

$$\sigma_x - \sigma_y = 2k \sin 2\gamma. \tag{2.5}$$

By use of (2.1), the equations (2.4), (2.5) become, respectively

$$\frac{\partial^2 F}{\partial x \partial y} = -k \cos 2\gamma, \tag{2.6}$$

$$\frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial x^2} = 2k \sin 2\gamma. \tag{2.7}$$

In our future work, we shall study the system (2.6), (2.7).

Now, we consider the boundary conditions. Let the stresses acting on that part of the boundary which is adjacent to the plastic material be

$$\sigma_x(0, y) = \phi(y), \qquad \sigma_{xy}(0, y) = \theta(y), \tag{2.8}$$

where $\phi(y)$ and $\theta(y)$ are arbitrary functions of y possessing piecewise continuous second derivatives. Note, the stress σ_y $(0, y)$ can be found in terms of $\phi(y)$ and $\theta(y)$ by use of (2.1), (2.6), (2.7). Further, by use of (2.1), (2.8), we find

$$\phi(y) = \frac{\partial^2 F}{\partial y^2}\Big)_{0,y}, \qquad \theta(y) = -\frac{\partial^2 F}{\partial x \partial y}\Big)_{0,y}. \tag{2.9}$$

By integrating the first equation of (2.9) twice with respect to y, we find

$$F(0, y) = \alpha(y) = \int_{y_0}^{y} dy \int_{y_0}^{y} \phi(y)\, dy. \tag{2.10}$$

Similarly, by integrating the second equation of (2.9) once with respect to y, we find

$$-\frac{\partial F}{\partial x}\Big)_{0,y} = \beta(y) = \int_{y_0}^{y} \theta(y)\, dy. \tag{2.11}$$

Equations (2.10), (2.11) furnish Cauchy data for the integration of (2.6) or (2.7). In view of the fact that the y-axis is a characteristic of (2.6), we see that for the above data only (2.7) may be integrated. After integrating (2.7), we shall require that $F(x, y)$ satisfy (2.6). This last condition will furnish a relation between the sine and cosine functions of the angle 2γ and our boundary data. Returning to the functions $\alpha(y)$ and $\beta(y)$, we note that $\alpha(y)$ is determined to within a linear function of y and $\beta(y)$ is determined to within an arbitrary constant. To obtain the function $F(x, y)$, we shall consider $\alpha(y)$ and $\beta(y)$ as completely determined. However, the calculation of the stresses requires that only the second derivative of $\alpha(y)$ and the first derivative of $\beta(y)$ be known. From (2.10), (2.11), we see that these derivatives are uniquely determined, that is

$$\frac{d^2 \alpha}{dy^2} = \phi(y), \qquad \frac{d\beta}{dy} = \theta(y). \tag{2.12}$$

3. **Integration.** First, we turn to the integration of (2.7). This equation is similar to the wave equation. Its integral is [13]

$$F(\xi, \eta) = \frac{1}{2}\left[\alpha(\eta - \xi) + \alpha(\eta + \xi) + \int_{\eta+\xi}^{\eta-\xi} \beta(y)\, dy\right] - k\iint_{\triangle} \sin 2\gamma\, dx\, dy. \tag{3.1}$$

In (3.1), the coordinates (ξ, η) are those of an arbitrary point P in the perfectly plastic region R, the multiple integral is taken over the triangle bounded by the y-axis and the characteristics through the point P. Note, the characteristics of (2.7) are straight lines inclined at 45° and 135° to the x-axis (see figure 1).

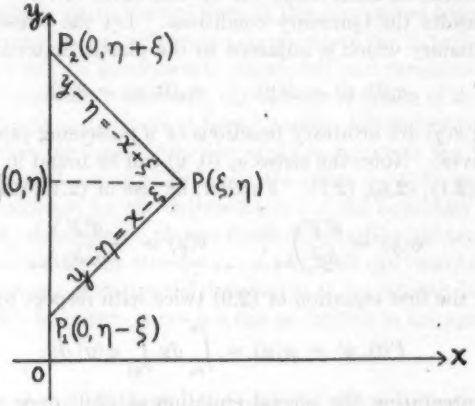

FIG. 1.

By use of (2.1), (3.1), we can calculate the stresses in terms of the known boundary data and the unknown function γ. Let us write

$$u = \eta - \xi, \qquad v = \eta + \xi. \tag{3.2}$$

Forming the second derivatives of (3.1), we find

$$\sigma_\xi = \frac{\partial^2 F}{\partial \eta^2} = \tfrac{1}{2}[\phi(u) + \phi(v) + \theta(u) - \theta(v)] - k\frac{\partial^2}{\partial \eta^2}\iint_{\triangle} \sin 2\gamma\, dx\, dy, \tag{3.3}$$

$$\sigma_\eta = \frac{\partial^2 F}{\partial \xi^2} = \tfrac{1}{2}[\phi(u) + \phi(v) + \theta(u) - \theta(v)] - k\frac{\partial^2}{\partial \xi^2}\iint_{\triangle} \sin 2\gamma\, dx\, dy, \tag{3.4}$$

$$\sigma_{\xi\eta} = -\frac{\partial^2 F}{\partial \xi \partial \eta} = \tfrac{1}{2}[\phi(u) - \phi(v) + \theta(u) + \theta(v)] + k\frac{\partial^2}{\partial \xi \partial \eta}\iint_{\triangle} \sin 2\gamma\, dx\, dy. \tag{3.5}$$

By substituting (3.5) into (2.6), we obtain an integral-differential equation for the determination of $\gamma(x, y)$ in terms of the boundary data

$$\cos 2\gamma = \frac{1}{2k}[\phi(u) - \phi(v) + \theta(u) + \theta(v)] + \frac{\partial^2}{\partial \xi \partial \eta}\iint_{\triangle} \sin 2\gamma\, dx\, dy. \tag{3.6}$$

In the next few paragraphs, we shall study the relation (3.6).

In view of the nature of our region of integration, it is easily shown that the differentiation operator in (3.6) may be taken inside the integral sign for those regions R in which $\sin 2\gamma$ and its first and second partial derivatives exist and are continuous. Let us assume that the function $\sin 2\gamma$ has these properties in R. Further, let us replace $\sin 2\gamma$ by $f(x, y)$ for convenience in writing. From figure 1, we see that

$$(3.7) \qquad \iint_{\triangleright} f(x, y)\, dx\, dy = \int_0^{\xi} dx \int_{\eta}^{\eta - x + \xi} f(x, y)\, dy + \int_0^{\xi} dx \int_{\eta + x - \xi}^{\eta} f(x, y)\, dy.$$

By differentiating the first integral in the right hand side of (3.7) partially with respect to η, we obtain

$$(3.8) \qquad \frac{\partial}{\partial \eta} \int_0^{\xi} dx \int_{\eta}^{\eta - x + \xi} f(x, y)\, dy = \int_0^{\xi} [f(x, \eta - x + \xi) - f(x, \eta)]\, dx.$$

Next, we differentiate (3.8) partially with respect to ξ and find

$$(3.9) \qquad \frac{\partial^2}{d\xi \partial \eta} \int_0^{\xi} dx \int_{\eta}^{\eta - x + \xi} f(x, y)\, dy = \int_0^{\xi} \frac{\partial f(x, \eta - x + \xi)}{\partial(\eta - x + \xi)}\, dx.$$

Upon replacing the variable of integration x in the right hand side of (3.9) by

$$(3.10) \qquad x = \xi - Z,$$

the equation (3.9) becomes

$$(3.11) \qquad \frac{\partial^2}{\partial \xi \partial \eta} \int_0^{\xi} dx \int_{\eta}^{\eta - x + \xi} f(x, y)\, dy = \int_0^{\xi} \frac{\partial f(\xi - Z, \eta + Z)}{\partial(\eta + Z)}\, dZ.$$

The integral in the right hand side of (3.11) is nothing but the line integral of $\partial f/\partial y$ along the characteristic PP_2 (see figure 1). By the further change of variable

$$(3.12) \qquad y = \eta + Z,$$

the equation (3.11) becomes

$$(3.13) \qquad \frac{\partial^2}{\partial \xi \partial \eta} \int_0^{\xi} dx \int_{\eta}^{\eta - x + \xi} f(x, y)\, dy = \int_P^{P_2} \frac{\partial f(x, y)}{\partial y}\, dy, \quad \text{along} \quad PP_2.$$

Similarly, we can show for the second integral in the right hand side of (3.7)

$$(3.14) \qquad \frac{\partial^2}{\partial \xi \partial \eta} \int_0^{\xi} dx \int_{\eta + x - \xi}^{\eta} f(x, y)\, dy = \int_{P_1}^{P} \frac{\partial f(x, y)}{\partial y}\, dy, \quad \text{along} \quad P_1P.$$

Thus, from (3.7), (3.13), (3.14), we find that

$$(3.15) \qquad \frac{\partial^2}{\partial \xi \partial \eta} \iint_{\triangleright} f(x, y)\, dx\, dy = \int_{P_1}^{P_2} \frac{\partial f(x, y)}{\partial y}\, dy, \quad \text{along} \quad P_1PP_2,$$

where the integral in the right hand side of (3.15) is a line integral along the characteristics in figure 1. Again, we can write (3.7) as

$$(3.16) \qquad \iint_{\triangleright} f(x, y)\, dx\, dy = \int_{\eta - \xi}^{\eta} dy \int_0^{y - \eta + \xi} f(x, y)\, dx + \int_{\eta}^{\eta + \xi} dy \int_0^{\xi - y + \eta} f(x, y)\, dx.$$

Proceeding as in (3.8) to (3.15), we obtain a formula analogous to (3.15), namely

$$\frac{\partial^2}{\partial\xi\partial\eta}\iint_{\triangleright} f(x,y)\,dx\,dy = -\int_{P_1}^{P_2}\frac{\partial f(x,y)}{\partial x}\,dx + f(0,\eta+\xi) - f(0,\eta-\xi), \tag{3.17}$$

where the integral in the right hand side of (3.17) is a line integral along P_1PP_2. Combining (3.15), (3.17), we find

$$\frac{\partial^2}{\partial\xi\partial\eta}\iint_{\triangleright} f(x,y)\,dx\,dy = -\frac{1}{2}\int_{P_1}^{P_2}\left(\frac{\partial f}{\partial x}\,dx - \frac{\partial f}{\partial y}\,dy\right) + \frac{f(0,\eta+\xi) - f(0,\eta-\xi)}{2}. \tag{3.18}$$

Let us replace the last two terms in the right hand side of (3.18) by

$$\frac{f(0,\eta+\xi) - f(0,\eta-\xi)}{2} = \frac{1}{2}\int_{P_2}^{P_1}\frac{\partial f}{\partial y}\,dy + f(0,\eta+\xi) - f(0,\eta-\xi), \tag{3.19}$$

where the integral in the right hand side is taken along the y-axis. By use of (3.19), the equation (3.18) becomes

$$\frac{\partial^2}{\partial\xi\partial\eta}\iint_{\triangleright} f(x,y)\,dx\,dy = -\frac{1}{2}\oint\left(\frac{\partial f}{\partial x}\,dx - \frac{\partial f}{\partial y}\,dy\right) + f(0,\eta+\xi) - f(0,\eta-\xi), \tag{3.20}$$

where the line integral is taken about the triangle of figure 1. Using Green's theorem in (3.20) and replacing $f(x,y)$ by $\sin 2\gamma$, we obtain

$$\frac{\partial^2}{\partial\xi\partial\eta}\iint_{\triangleright}\sin 2\gamma\,dx\,dy = \iint_{\triangleright}\frac{\partial^2\sin 2\gamma}{\partial x\partial y}\,dx\,dy + \sin 2\gamma\Big)_{0,\eta+\xi} - \sin 2\gamma\Big)_{0,\eta-\xi}. \tag{3.21}$$

From (2.1), (2.4), (2.9), we find the value of $\cos 2\gamma$ along the y-axis, that is

$$\cos 2\gamma\Big)_{0,y} = -\theta(y)/k. \tag{3.22}$$

Hence, substituting (3.21), (3.22) into (3.6), we obtain with the aid of (3.2)

$$\cos 2\gamma = \frac{1}{2k}[\phi(u) - \phi(v) + \theta(u) + \theta(v) + 2\sqrt{k^2-\theta^2(v)} - 2\sqrt{k^2-\theta^2(u)}] + \iint_{\triangleright}\frac{\partial^2\sin 2\gamma}{\partial x\partial y}\,dx\,dy. \tag{3.23}$$

Our next step is to apply the operator

$$D^2 \equiv \frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\eta^2} \tag{3.24}$$

to both sides of (3.23). A calculation similar to that used in the preceeding paragraph reveals that

$$D^2\iint_{\triangleright}\frac{\partial^2\sin 2\gamma}{\partial x\partial y}\,dx\,dy = 2\,\frac{\partial^2\sin 2\gamma}{\partial x\partial y}. \tag{3.25}$$

Further, in view of the fact that u and v are variables of the form given by (3.2), it follows that

$$D^2 f(u) = D^2 g(v) = 0, \tag{3.26}$$

where f and g are arbitrary functions (possessing piecewise continuous second derivatives) of the variables u and v, respectively. Hence, the application of the operator (3.24) to (3.23) furnishes[4]

$$\left(\frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \eta^2}\right) \cos 2\gamma = 2 \frac{\partial^2}{\partial \xi \partial \eta} \sin 2\gamma. \tag{3.27}$$

Substituting (3.27) into (3.23), the latter becomes

$$\begin{aligned} \cos 2\gamma = {} & \frac{1}{2k}[\phi(u) - \phi(v) + \theta(u) + \theta(v) + 2\sqrt{k^2 - \theta^2(v)} - 2\sqrt{k^2 - \theta^2(u)}] \\ & + \frac{1}{2} \iint_{\triangleright} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) \cos 2\gamma \, dx \, dy. \end{aligned} \tag{3.28}$$

In view of (3.26), it is seen by direct substitution that the solution of (3.28) is

$$\cos 2\gamma = \frac{1}{2k}[\phi(u) - \phi(v) + \theta(u) + \theta(v) + 2\sqrt{k^2 - \theta^2(v)} - 2\sqrt{k^2 - \theta^2(u)}]. \tag{3.29}$$

We shall now show that the locus

$$\cos 2\gamma = \pm 1, \tag{3.30}$$

is a line of rupture in the sense of Christianovich [2]. Let us for convenience in future writing set

$$\cos 2\gamma = G(\xi, \eta). \tag{3.31}$$

From (3.31), it follows that

$$\sin 2\gamma = \pm\sqrt{1 - G^2(\xi, \eta)}. \tag{3.32}$$

In those regions R, where $(1 - G^2)^{1/2}$ is continuous and possesses continuous first and second partial derivatives, we find by arguments similar to those used in deriving (3.13), (3.14)

$$\frac{\partial^2}{\partial \xi^2} \iint_{\triangleright} \sqrt{1 - G^2} \, dx \, dy = -2G(1 - G^2)^{-1/2} \frac{\partial G}{\partial \xi} + \int_{P_1}^{P_2} \frac{\partial^2 \sqrt{1 - G^2}}{\partial y^2} \, dy, \tag{3.33}$$

where the integral in (3.33) is a line integral along P_1PP_2. If we allow P to approach a point $\bar{P}$ which lies on the locus (3.30) and if $\partial G/\partial \xi$ does not vanish in that portion of R under discussion, then the left hand side of (3.33) becomes unbounded. If $\partial G/\partial \xi$ does vanish then G is a function of η alone in this portion of R. Hence, (3.33) becomes

$$\begin{aligned} \frac{\partial^2}{\partial \xi^2} \iint_{\triangleright} \sqrt{1 - G^2} \, dx \, dy = {} & -\int_{P_1}^{P} d\left[G(1 - G^2)^{-1/2} \frac{dG}{\partial y}\right] \\ & + \int_{P_2}^{P} d\left[G(1 - G^2)^{-1/2} \frac{dG}{dy}\right]. \end{aligned} \tag{3.34}$$

[4] Relation (3.27) follows directly if (2.6), (2.7) are differentiated.

As P approaches $\bar{P}$, the integrals in the right hand side of (3.34) become divergent. Hence, again the left hand side of (3.34) becomes unbounded. Thus, we see by use of (3.32) that the quantity

$$\frac{\partial^2}{\partial \xi^2} \iint_{\triangle} \sin 2\gamma \, dx \, dy \tag{3.35}$$

becomes unbounded as P approaches $\bar{P}$. Finally, from (3.4), it follows that $\partial \sigma_n / \partial \xi$ becomes unbounded as P approaches $\bar{P}$. Thus, the locus (3.30) represents a line of rupture in the sense of Christianovich. That is, the locus (3.30) represents a "curve beyond which the statically determined plastic stress distribution cannot be extended."[2]

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AN IMPROVEMENT ON THE G.C.D. METHOD FOR COMPLEX ROOTS

By Frank L. Hitchcock

The method of Dr. Shih-Nge Lin, when it converges rapidly, is the most satisfactory yet devised for finding complex roots of algebraic equations with real coefficients, (1), (2). For equations of the fourth degree the method of Henry S. Sharp (3) is rapid and easy and applies to all such equations, no matter how nearly equal the two real quadratic factors may be. The G.C.D. method (4), (5) can be systematically applied to equations of any degree but leaves something to be desired in the way of speed. The following plan offers a distinct improvement.

Writing the given equation as

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0 \tag{1}$$

the coefficients being real, if we divide $P(x)$ by the real quadratic $x^2 + px + q$ we shall have a remainder $r_1x + r_0$ and a quotient $Q(x)$ of degree $n - 2$. Now divide this quotient by the original divisor $x^2 + px + q$ and get a second remainder $u_1x + u_0$. Next compute the two new quantities

$$v_1 = u_0 - pu_1, \qquad v_0 = -qu_1 \tag{2}$$

and then take

$$p' = \frac{r_1u_0 - r_0u_1}{v_1u_0 - v_0u_1}, \qquad q' = \frac{-r_1v_0 + r_0v_1}{v_1u_0 - v_0u_1}. \tag{3}$$

If $x^2 + px + q$ was approximately a factor of $P(x)$ then

$$x^2 + (p + p')x + q + q' \tag{4}$$

will be a better approximation to a true factor, and often a much better approximation.

We then repeat the process, starting with (4) as a new divisor. All steps are readily carried out on a computing machine.

I shall next illustrate by an example and afterwards show why the process is in general valid.

Take the following equation, which has been studied both by Dr. Lin and the author.

$$x^8 - 3.012x^7 + 3.225x^6 + 1.021x^5 + 6.986x^4 - 21.887x^3 + 8.110x^2 + 5.901x + 23.889 = 0 \tag{5}$$

Let it be known that $x^2 + x + 1$ is, roughly, a factor. (The important question how to find such a factor will be discussed later.) Dividing the left side by $x^2 + x + 1$ we find a remainder $-0.256x - 4.904$, that is

$$r_1 = -0.256, \qquad r_0 = -4.904. \tag{6}$$

The quotient in this division was

$$Q(x) = x^6 - 4.012x^5 + 6.237x^4 - 1.204x^3 + 1.953x^2 - 22.636x + 28.793. \quad (7)$$

Dividing $Q(x)$ by $x^2 + x + 1$ we find the second remainder $-13.340x + 30.738$, that is

$$u_1 = -13.340, \quad u_0 = 30.738. \quad (8)$$

By (2) we find, since $p = q = 1$,

$$v_1 = 44.078, \quad v_0 = 13.440 \quad (9)$$

Then by (3)

$$p' = -.0478, \quad q' = -.1388. \quad (10)$$

Taking these as having some significance to two decimal places we put

$$p + p' = 0.95, \quad q + q' = 0.86 \quad (11)$$

and so take $x^2 + 0.95x + 0.86$ as a new divisor.

That is to say, we start fresh, taking $p = 0.95$ and $q = 0.86$, and divide the original $P(x)$ by $x^2 + 0.95x + 0.86$. This gives a quotient

$$Q(x) = x^6 - 3.962x^5 + 6.1289x^4 - 1.394135x^3 + 3.03957425x^2 - 23.57563944x + 27.89282361 \quad (12)$$

and a remainder

$$-0.32213251x - 0.09882830 = r_1x + r_0. \quad (13)$$

The division of $Q(x)$ by $x^2 + 0.95x + 0.86$ gives a second remainder

$$-18.62694975x + 27.22791569 = u_1x + u_0 \quad (14)$$

whence we find

$$p' = -.00697432, \quad q' = +.00047358 \quad (15)$$

whence

$$p + p' = .94302568, \quad q + q' = .86047358. \quad (16)$$

Judging it likely that these results may have significance to five places we now take $x^2 + .94303x + .86047$ as the next divisor. The process was carried through once more with this last quadratic as $x^2 + px + q$ and new corrections p' and q' obtained, yielding a corrected divisor.

$$x^2 + .94301\ 66365x + .86047\ 37372 \quad (17)$$

in which the tenth decimal place is in doubt.

In this example, at least, it is evident that the convergence is rapid, yielding successively two, five, and nine significant figures, in the three repetitions of the process.

Theory of the Method. In the first division the quotient was Q and the remainder $r_1x + r_0$,

$$P = Q(x^2 + px + q) + r_1x + r_0. \tag{18}$$

When Q is divided by $x^2 + px + q$ let the quotient be called Q',

$$Q = Q'(x^2 + px + q) + u_1x + u_0 \tag{19}$$

which substituted in (18) gives

$$P = Q'(x^2 + px + q)^2 + (u_1x + u_0)(x^2 + px + q) + r_1x + r_0. \tag{20}$$

This is an identity in x, p, q, and the coefficients a_0, a_1, etc., the four quantities u_1, u_0, r_1, and r_0 being polynomials in p, q, and the a's. Take the partial derivative of both sides of this identity with respect to q,

$$0 = \frac{\partial Q'}{\partial q}(x^2 + px + q)^2 + 2Q'(x^2 + px + q) + \left(x\frac{\partial u_1}{\partial q} + \frac{\partial u_0}{\partial q}\right)(x^2 + px + q) + (u_1x + u_0) + x\frac{\partial r_1}{\partial q} + \frac{\partial r_0}{\partial q}. \tag{21}$$

Since this relation is also an identity it will hold true when we substitute for x a root of the equation $x^2 + px + q = 0$, say $x = a + bj$ where $j^2 = -1$. This gives

$$0 = u_1(a + bj) + u_0 + (a + bj)\frac{\partial r_1}{\partial q} + \frac{\partial r_0}{\partial q}, \tag{22}$$

and by equating real and imaginary terms separately to zero

$$u_1 = -\frac{\partial r_1}{\partial q}, \qquad u_0 = -\frac{\partial r_0}{\partial q}, \tag{23}$$

that is, the second remainder $u_1x + u_0$ is the negative of the derivative of the first remainder with respect to q.

We next take the partial derivative of (20) with respect to p.

$$0 = \frac{\partial Q'}{\partial p}(x^2 + px + q)^2 + 2xQ'(x^2 + px + q) + \left(x\frac{\partial u_1}{\partial p} + \frac{\partial u_0}{\partial p}\right)(x^2 + px + q) + x(u_1x + u_0) + x\frac{\partial r_1}{\partial p} + \frac{\partial r_0}{\partial p}. \tag{24}$$

As before we now set $x = a + bj$, a root of $x^2 + px + q = 0$, yielding

$$0 = (a + bj)(u_1a + u_1bj + u_0) + (a + bj)\frac{\partial r_1}{\partial p} + \frac{\partial r_0}{\partial p}. \tag{25}$$

Equating real and imaginary parts to zero,

$$(a^2 - b^2)u_1 + au_0 + a\frac{\partial r_1}{\partial p} + \frac{\partial r_0}{\partial p} = 0 \tag{26}$$

$$2au_1 + u_0 + \frac{\partial r_1}{\partial p} = 0. \tag{27}$$

From (27), since $2a = -p$, we find

$$\frac{\partial r_1}{\partial p} = pu_1 - u_0 = -v_1 \tag{28}$$

and then, by (26), using $q = a^2 + b^2$,

$$\frac{\partial r_0}{\partial p} = qu_1 = -v_0; \tag{29}$$

thus v_1 and v_0 are the negatives of the derivatives of r_1 and r_0 with respect to p.

Since r_1 and r_0 are polynomials in p and q we may expand them by Taylor's series; we assume p' and q' small enough so that their squares may be neglected, and so chosen as to make these remainders vanish. Then

$$r_1(p + p', q + q') = r_1(p, q) + p' \frac{\partial r_1}{\partial p} + q' \frac{\partial r_1}{\partial q} = 0 \tag{30}$$

$$r_0(p + p', q + q') = r_0(p, q) + p' \frac{\partial r_1}{\partial p} + q' \frac{\partial r_0}{\partial q} = 0. \tag{31}$$

By (28) and (29) and (23) these results become

$$p'v_1 + q'u_1 = r_1, \qquad p'v_0 + q'u_0 = r_0 \tag{32}$$

which solved for p' and q' give (3).

Discussion. Compared with the G.C.D. method of the paper (4), the above process replaces the labor of subtracting, by Horner's method, the real part of the approximate root, by the smaller labor of two successive divisions. Moreover the computation of p' and q' by the formulas (2) and (3) of the present paper is much quicker than that of s, t, u, and v of the former method. The rapidity of the convergence at successive approximations appears to be at least as good.

It may be worth while to remark that, in dividing $P(x)$ by $x^2 + px + q$ on a computing machine, only the successive coefficients of the quotient, and, of course, the remainder, need be written down, all intermediate steps being carried on the machine. If three successive coefficients in $Q(x)$ are q_k, q_{k-1}, and q_{k-2} we have in general

$$q_{k-2} = a_k - qq_k - pq_{k-1}. \tag{33}$$

Furthermore two successive divisions are actually to be carried along together. For while q_{k-2} is still on the machine we compute the corresponding term of the second quotient $Q'(x)$ in a similar manner.

It is of interest to compare the present method with that of Dr. Lin in a case where Dr. Lin's method converges rather slowly. In his example (b), page 76 of paper (2) he obtains the factor

$$x^2 - 0.821x + 0.739 \tag{34}$$

by five repetitions. Using the same initial divisor as in his method, namely

$$x^2 - 0.82x + 0.86 \tag{35}$$

two repetitions of the present method yielded the factor

$$x^2 - 0.8210x + 0.7378 \tag{35}$$

which are exact to four decimal places. One repetition of the present method requires two divisions and the computation of p and p'. If only two or three decimals are needed the present method does not seem to have much advantage if Dr. Lin's method converges fairly well. If greater accuracy is required the rapid convergence of the present method would yield considerable advantage.

In judging of the accuracy attained at any approximation, we note that p' and q' are obtained on the supposition that their squares are negligible. Accordingly when p' and q' were found in (10) it appeared that not more than two decimal places were worth keeping; similarly in (15) since p'^2 is nearly .00005 not more than five places are likely to be of value; finally in (17), since p'^2 is more than .0000 0000 02, we conclude the tenth place is in doubt.

The first approximation to a quadratic factor. The process above illustrated appears satisfactory for improving an approximate quadratic factor already found but tells nothing about how to obtain the first approximation. When Dr. Lin's method fails to converge the initial quadratic determined by his rule is likely not to give convergence under the present method either. The G.C.D. can in general be used. It consists, in principle, of moving the origin until we have an approximate quadratic factor of the form $x^2 + q$, that is, with $p = 0$. But the work is rather lengthy.

In many cases a few trial divisions of $P(x)$ by $x^2 + px + q$ with p and q taken as whole numbers leads rapidly to a good approximate quadratic factor. In the first illustrative example above, the divisor $x^2 + x + 1$ was actually obtained as follows. Divisions were made by the trial factors x^2, $x^2 + x$, $x^2 + 1$, and $x^2 + x + 1$ which do not take long. We tabulate the remainders,

p	q	r_1	r_0	
0	0	5.901	23.889	
1	0	−37.298	23.889	(36)
0	1	+31.821	20.540	
1	1	−0.256	−4.904	

The changes of sign indicate the existence of an exact divisor with both p and q less than $+1$. The smallness of the remainder suggested trying $x^2 + x + 1$.

If we regard p and q as rectangular coordinates in a pq-plane, the equations $r_1 = 0$ and $r_0 = 0$ define two algebraic curves in this plane. Each division by $x^2 + px + q$, with p and q integers, determines the values of r_1 and of r_0 at a point in this plane, at the corner of a unit square. Moving from point to point

in the pq-plane, the changes in sign of r_1 and r_0 are a guide as to the next trial divisor. An interpolation sometimes helps, and can be carried out graphically. With the data of (36), by interpolating between (0, 0) and (1, 0), we say $r_1 = 0$ at the point (.14, 0); and by interpolating between (0, 1) and (1, 1) we sav $r_1 = 0$ at the point (.99, 1). We join these two points by a straight line. Again. interpolating between (0, 1) and (1, 1) we say $r_0 = 0$ at the point (.8, 1); and interpolating between (1, 0) and (1, 1) we say $r_0 = 0$ at (1, .82). We join these two points by a straight line. The two lines intersect nearly at (.9, .9), suggesting $x^2 + .9x + .9$ as initial divisor, which, since the true factor turned out to be about $x^2 + .94x + .86$, is distinctly better than $x^2 + x + 1$.

If the roots of $x^2 + px + q$ are complex imaginaries the point (p, q) in the pq-plane will lie above the parabola $p^2 - 4q = 0$. And if $P(x)$ is of even degree with roots imaginary the absolute term a_0 will be the product of the absolute values of all the roots. Any real quadratic factor $x^2 + px + q$ will have the value of q equal to the product of the absolute values of one pair of roots. Hence $P(x)$ has at least one factor $x^2 + px + q$ such that q is not greater than the m^{th} root of a_0, where $2m = n$, the degree of $P(x)$. Therefore in mapping the pq-plane according to the above method we need map only the region above the parabola $p^2 - 4q = 0$ and below the straight line $q = \sqrt[m]{a_0}$ in order to find at least one quadratic factor.

It may be further noticed that there is nothing in the above methods which prevents their being applied to finding real quadratic factors of any equation with real coefficients, whether the roots are complex or not. For the identities (23), (28), and (29) which were proved to hold whenever the roots of $x^2 + px + q$ were complex imaginaries, that is over a continuous region in the pq-plane, must hold for all values of p and q whatever, since these are merely identities between polynomials in p and q.

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TABLE OF COEFFICIENTS FOR INVERSE INTERPOLATION WITH ADVANCING DIFFERENCES

By Herbert E. Salzer*

This table might be regarded as a companion table (as well as a supplement) to the one previously published in the article "Table of Coefficients for Inverse Interpolation with Central Differences", Jour. of Math. and Phys., vol. XXII, No. 4, Dec. 1943, pp. 210–224. There are many occasions in the performance of inverse interpolation where only advancing (or backward) differences are available. Examples occur when the inverse interpolation is at the beginning (or end) of a table and also when only a few values are tabulated altogether (which often occurs in finding the zeros of involved expressions). Thus in a strict sense, because this table can be used everywhere, it is even more fundamental than the preceding one which employs central differences.

The polynomial expressions for the coefficients are given in H. T. Davis' "Tables of the Higher Mathematical Functions," vol. I, pp. 80–81. Davis applied Lagrange's theorem on inversion to the Gregory-Newton formula for direct interpolation. The formula obtained by Davis was checked and found to be entirely correct except for two obvious printing errors, namely, p. 81 line 6 for $3!^2$ read $4!^2$ and on line 7 for $+36$ read -36. In addition, it was thought desirable by the author to add to the formula the coefficients of $\left(\frac{\Delta^2}{\Delta}\right)^4$ and $\frac{\Delta^2}{\Delta}\frac{\Delta^6}{\Delta}$.

When there is a uniform interval h for the argument x, so that $x_1 - x_0 = h$, given $y = f(x)$, the problem is to find p such that $x = x_0 + ph$. We restrict the problem to where $f(x)$ is monotonic between x_0 and x_1 so that if m denotes $\frac{y - y_0}{y_1 - y_0}$ or $\frac{y - y_0}{\Delta}$, then $0 < m < 1$. Using the notation Δ_x^1 (or Δ) $= f(x + h) - f(x)$ and $\Delta_x^r = \Delta_{x+h}^{r-1} - \Delta_x^{r-1}$ and omitting the subscript x for convenience, Davis' formula (with a few terms omitted because they are relatively very small and with the slight addition mentioned above) reads as follows:

$$p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta} + \frac{m(m-1)(2m-1)}{4}\left(\frac{\Delta^2}{\Delta}\right)^2$$
$$- \frac{m(m-1)(m-2)(m-3)}{24}\frac{\Delta^4}{\Delta} + \frac{m(m-1)(5m^2-11m+4)}{12}\frac{\Delta^2}{\Delta}\frac{\Delta^3}{\Delta}$$
$$- \frac{m(m-1)(m-2)(m-3)(m-4)}{120}\frac{\Delta^5}{\Delta} - \frac{m(m-1)(5m^2-5m+1)}{8}\left(\frac{\Delta^2}{\Delta}\right)^3$$
$$+ \frac{m(m-1)(m-2)(3m^2-6m+2)}{36}\left(\frac{\Delta^2}{\Delta}\right)^3$$

* Mathematical Tables Project, National Bureau of Standards.

$$+\frac{m(m-1)(6m^3-29m^2+39m-12)}{48}\frac{\Delta^3}{\Delta}\frac{\Delta^4}{\Delta}$$

$$-\frac{m(m-1)(m-2)(m-3)(m-4)(m-5)}{720}\frac{\Delta^6}{\Delta}$$

$$-\frac{m(m-1)(21m^3-54m^2+36m-6)}{24}\left(\frac{\Delta^3}{\Delta}\right)^2\frac{\Delta^3}{\Delta}$$

$$+\frac{m(m-1)(m-2)(7m^3-33m^2+42m-12)}{144}\frac{\Delta^3}{\Delta}\frac{\Delta^4}{\Delta}$$

$$+\frac{m(m-1)(7m^4-59m^3+166m^2-174m+48)}{240}\frac{\Delta^2}{\Delta}\frac{\Delta^5}{\Delta}$$

$$-\frac{m(m-1)(m-2)(m-3)(m-4)(m-5)(m-6)}{5040}\frac{\Delta^7}{\Delta}$$

$$+\frac{m(m-1)(14m^3-21m^2+9m-1)}{16}\left(\frac{\Delta^2}{\Delta}\right)^4$$

$$-\frac{m(m-1)(28m^4-119m^3+166m^2-84m+12)}{72}\frac{\Delta^2}{\Delta}\left(\frac{\Delta^3}{\Delta}\right)^2$$

$$-\frac{m(m-1)(28m^4-140m^3+220m^2-120m+18)}{96}\left(\frac{\Delta^2}{\Delta}\right)^2\frac{\Delta^4}{\Delta}$$

$$+\frac{m(m-1)(m-2)(m-3)(4m^3-18m^2+22m-6)}{576}\left(\frac{\Delta^4}{\Delta}\right)^2$$

$$+\frac{m(m-1)(m-2)(8m^4-67m^3+185m^2-186m+48)}{720}\frac{\Delta^3}{\Delta}\frac{\Delta^5}{\Delta}$$

$$+\frac{m(m-1)(4m^5-52m^4+248m^3-527m^2+471m-120)}{720}\frac{\Delta^2}{\Delta}\frac{\Delta^6}{\Delta}$$

$$-\frac{m(m-1)(m-2)(m-3)(m-4)(m-5)(m-6)(m-7)}{40320}\frac{\Delta^8}{\Delta}+\cdots$$

We shall refer to the coefficients of $\left(\frac{\Delta^i}{\Delta}\right)^r\cdot\left(\frac{\Delta^j}{\Delta}\right)^s\cdots$ as belonging to terms of the $[ir+sj+\cdots]$th order. Then it is apparent that the preceding formula includes all terms through the eighth order, or, more specifically, it contains one second order term, one third order term, two fourth order terms, two fifth order terms, four sixth order terms, four seventh order terms and seven eighth order terms. The following tables give the fourth and fifth order terms for $m = 0(.001)1.000$, the sixth order terms for $m = 0(.01)1.00$ and the seventh and eighth order terms for $m = 0(.1)1.0$. All these values are correct to about half a unit in the tenth decimal place with the exception of the coefficient of $\frac{\Delta^2}{\Delta}\frac{\Delta^3}{\Delta}$ which is accurate to within a unit in the tenth decimal place. A condensed one-page table of coefficients of the fourth, fifth and sixth order terms, at inter-

vals of 0.1, is also given for the sake of convenience, since there are many cases where a larger table is unnecessary.

There was no need to tabulate the coefficients of the second and third order terms, $\frac{m(1-m)}{2}$ and $\frac{m(1-m)(m-2)}{6}$ respectively, since they are both tabulated at intervals of 0.0001 by the Mathematical Tables Project in the Table of Lagrangian Interpolation Coefficients. For m (there "p") ranging from 0 to 1, $\frac{m(1-m)}{2}$ is given by $-A_{-1}$ in the three-point table and $\frac{m(1-m)(m-2)}{6}$ is given by A_{-1} in the four-point table. Linear interpolation in those tables will furnish 9 decimals. However, it is probably most expedient, especially for high accuracy, to compute $\frac{m(1-m)}{2}$ on a good calculating machine and then to multiply the result by $\frac{m-2}{3}$, thus obtaining the coefficients for both $\frac{\Delta^2}{\Delta}$ and $\frac{\Delta^3}{\Delta}$.

Upper bounds for the error in x, due to the neglect of various terms in this formula, are obtained by multiplying h times the absolute values of $\left(\frac{\Delta^i}{\Delta}\right)^r \left(\frac{\Delta^j}{\Delta}\right)^s \cdots$ by corresponding upper bounds for the absolute values of the coefficients and then adding. The upper bounds for the absolute values of the various coefficients are as follows:

Order	Difference Function	Upper Bound for Coefficient
2nd	$\frac{\Delta^2}{\Delta}$	.125
3rd	$\frac{\Delta^3}{\Delta}$	.065
4th	$\left(\frac{\Delta^2}{\Delta}\right)^2$	.025
	$\frac{\Delta^4}{\Delta}$	.042

Order	Difference Function	Upper Bound for Coefficient
5th	$\frac{\Delta^2}{\Delta}\frac{\Delta^3}{\Delta}$	.027
	$\frac{\Delta^5}{\Delta}$	.031
6th	$\left(\frac{\Delta^3}{\Delta}\right)^2$	.0079
	$\left(\frac{\Delta^3}{\Delta}\right)^2$	.0077
	$\frac{\Delta^2}{\Delta}\frac{\Delta^4}{\Delta}$	.019
	$\frac{\Delta^6}{\Delta}$	.024
7th	$\left(\frac{\Delta^3}{\Delta}\right)^2\frac{\Delta^3}{\Delta}$	.015
	$\frac{\Delta^3}{\Delta}\frac{\Delta^4}{\Delta}$	.01
	$\frac{\Delta^2}{\Delta}\frac{\Delta^5}{\Delta}$	.015
	$\frac{\Delta^7}{\Delta}$	.02

approximately

Order	Difference Function	Upper Bound for Coefficients
8th	$\left(\frac{\Delta^3}{\Delta}\right)^4$	.003
	$\frac{\Delta^2}{\Delta}\left(\frac{\Delta^3}{\Delta}\right)^2$	.007
	$\left(\frac{\Delta^3}{\Delta}\right)^2\frac{\Delta^4}{\Delta}$	.008
	$\left(\frac{\Delta^4}{\Delta}\right)^2$	.004
	$\frac{\Delta^3}{\Delta}\frac{\Delta^5}{\Delta}$	.008
	$\frac{\Delta^2}{\Delta}\frac{\Delta^6}{\Delta}$	.012
	$\frac{\Delta^8}{\Delta}$	.017

approximately

TABLE 1A: SHORT TABLE OF COEFFICIENTS OF FOURTH, FIFTH, AND SIXTH ORDER TERMS

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta} + A(m)\left(\frac{\Delta^3}{\Delta}\right)^2 + B(m)\frac{\Delta^4}{\Delta} + C(m)\frac{\Delta^2}{\Delta}\frac{\Delta^3}{\Delta} + D(m)\frac{\Delta^5}{\Delta} + E(m)\left(\frac{\Delta^3}{\Delta}\right)^3 + F(m)\left(\frac{\Delta^3}{\Delta}\right)^2\frac{\Delta^4}{\Delta} + G(m)\frac{\Delta^3}{\Delta}\frac{\Delta^4}{\Delta} + H(m)\frac{\Delta^6}{\Delta}$$

Fourth Order			Fifth Order		
m	$A(m)$	$B(m)$	m	$C(m)$	$D(m)$
0	.000	.00000 00	0	.00000 000	.00000 000
.1	.018	.02066 25	.1	−.02212 500	−.01611 675
.2	.024	.03360 00	.2	−.02666 667	−.02553 600
.3	.021	.04016 25	.3	−.02012 500	−.02972 025
.4	.012	.04160 00	.4	−.00800 000	−.02995 200
.5	.000	.03906 25	.5	.00520 833	−.02734 375
.6	−.012	.03360 00	.6	.01600 000	−.02284 800
.7	−.021	.02616 25	.7	.02187 500	−.01726 725
.8	−.024	.01760 00	.8	.02133 333	−.01126 400
.9	−.018	.00866 25	.9	.01387 500	−.00537 075
1.0	.000	.00000 00	1.0	.00000 000	.00000 000

Sixth Order					
m	$E(m)$	$F(m)$	$G(m)$	$H(m)$	m
0	.00000 00	.00000 000	.00000 000	.00000 000	0
.1	.00618 75	.00679 250	.01572 000	.01316 201	.1
.2	.00400 00	.00736 000	.01770 667	.02042 880	.2
.3	−.00131 25	.00466 083	.01202 250	.02328 086	.3
.4	−.00600 00	.00085 333	.00328 000	.02296 320	.4
.5	−.00781 25	−.00260 417	−.00520 833	.02050 781	.5
.6	−.00600 00	−.00485 333	−.01128 000	.01675 520	.6
.7	−.00131 25	−.00553 583	−.01377 250	.01237 486	.7
.8	.00400 00	−.00469 333	−.01237 333	.00788 480	.8
.9	.00618 75	−.00266 750	−.00747 000	.00367 001	.9
1.0	.00000 00	.00000 000	.00000 000	.00000 000	1.0

TABLE I: COEFFICIENTS OF FOURTH ORDER TERMS

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta}$$

$$+ A(m)\left(\frac{\Delta^2}{\Delta}\right)^2 + B(m)\frac{\Delta^4}{\Delta} + \text{5th order terms} + \text{6th order terms}$$

$$+ \text{7th order terms} + \text{8th order terms} + \ldots$$

m	$A(m)$	$B(m)$		m
.000	.00000 00000	.00000 00000	.00000 00000	1.000
.001	.00024 92505	.00024 95419	.00008 33749	.999
.002	.00049 70040	.00049 81687	.00016 68327	.998
.003	.00074 32635	.00074 58817	.00025 03727	.997
.004	.00098 80320	.00099 26827	.00033 39947	.996
.005	.00123 13125	.00123 85729	.00041 76979	.995
.006	.00147 31080	.00148 35539	.00050 14819	.994
.007	.00171 34215	.00172 76273	.00058 53463	.993
.008	.00195 22560	.00197 07945	.00066 92905	.992
.009	.00218 96145	.00221 30570	.00075 33140	.991
.010	.00242 55000	.00245 44162	.00083 74162	.990
.011	.00265 99155	.00269 48738	.00092 15968	.989
.012	.00289 28640	.00293 44311	.00100 58551	.988
.013	.00312 43485	.00317 30897	.00109 01907	.987
.014	.00335 43720	.00341 08511	.00117 46031	.986
.015	.00358 29375	.00364 77166	.00125 90916	.985
.016	.00381 00480	.00388 36879	.00134 36559	.984
.017	.00403 57065	.00411 87664	.00142 82954	.983
.018	.00425 99160	.00435 29536	.00151 30096	.982
.019	.00448 26795	.00458 62510	.00159 77980	.981
.020	.00470 40000	.00481 86600	.00168 26600	.980
.021	.00492 38805	.00505 01821	.00176 75951	.979
.022	.00514 23240	.00528 08189	.00185 26029	.978
.023	.00535 93335	.00551 05718	.00193 76828	.977
.024	.00557 49120	.00573 94422	.00202 28342	.976
.025	.00578 90625	.00596 74316	.00210 80566	.975
.026	.00600 17880	.00619 45416	.00219 33496	.974
.027	.00621 30915	.00642 07736	.00227 87126	.973
.028	.00642 29760	.00664 61291	.00236 41451	.972
.029	.00663 14445	.00687 06094	.00244 96464	.971
.030	.00683 85000	.00709 42162	.00253 52162	.970
.031	.00704 41455	.00731 69509	.00262 08539	.969
.032	.00724 83840	.00753 88150	.00270 65590	.968
.033	.00745 12185	.00775 98098	.00279 23308	.967
.034	.00765 26520	.00797 99370	.00287 81690	.966
.035	.00785 26875	.00819 91979	.00296 40729	.965
.036	.00805 13280	.00841 75940	.00305 00420	.964
.037	.00824 85765	.00863 51268	.00313 60758	.963
.038	.00844 44360	.00885 17978	.00322 21738	.962
.039	.00863 89095	.00906 76084	.00330 83354	.961
.040	.00883 20000	.00928 25600	.00339 45600	.960
.041	.00902 37105	.00949 66542	.00348 08472	.959
.042	.00921 40440	.00970 98923	.00356 71963	.958
.043	.00940 30035	.00992 22760	.00365 36070	.957
.044	.00959 05920	.01013 38065	.00374 00785	.956
.045	.00977 68125	.01034 44854	.00382 66104	.955
.046	.00996 16680	.01055 43141	.00391 32021	.954
.047	.01014 51615	.01076 32941	.00399 98531	.953
.048	.01032 72960	.01097 14268	.00408 65628	.952
.049	.01050 80745	.01117 87137	.00417 33307	.951
.050	.01068 75000	.01138 51562	.00426 01562	.950
	$-A(m)$		$B(m)$	m

TABLE I: COEFFICIENTS OF FOURTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta}$$

$$+ A(m)\left(\frac{\Delta^2}{\Delta}\right)^2 + B(m)\frac{\Delta^4}{\Delta} + \text{5th order terms} + \text{6th order terms}$$

$$+ \text{7th order terms} + \text{8th order terms} + \ldots$$

m	$A(m)$	$B(m)$		
.050	.01068 75000	.01138 51562	.00426 01562	.950
.051	.01086 55755	.01159 07559	.00434 70389	.949
.052	.01104 23040	.01179 55140	.00443 39780	.948
.053	.01121 76885	.01199 94321	.00452 09731	.947
.054	.01139 17320	.01220 25117	.00460 80237	.946
.055	.01156 44375	.01240 47541	.00469 51291	.945
.056	.01173 58080	.01260 61609	.00478 22889	.944
.057	.01190 58465	.01280 67334	.00486 95024	.943
.058	.01207 45560	.01300 64731	.00495 67691	.942
.059	.01224 19395	.01320 53815	.00504 40885	.941
.060	.01240 80000	.01340 34600	.00513 14600	.940
.061	.01257 27405	.01360 07100	.00521 88830	.939
.062	.01273 61640	.01379 71330	.00530 63570	.938
.063	.01289 82735	.01399 27304	.00539 38814	.937
.064	.01305 90720	.01418 75036	.00548 14556	.936
.065	.01321 85625	.01438 14541	.00556 90791	.935
.066	.01337 67480	.01457 45834	.00565 67514	.934
.067	.01353 36315	.01476 68928	.00574 44718	.933
.068	.01368 92160	.01495 83838	.00583 22398	.932
.069	.01384 35045	.01514 90578	.00592 00548	.931
.070	.01399 65000	.01533 89162	.00600 79162	.930
.071	.01414 82055	.01552 79606	.00609 58236	.929
.072	.01429 86240	.01571 61923	.00618 37763	.928
.073	.01444 77585	.01590 36127	.00627 17737	.927
.074	.01459 56120	.01609 02232	.00635 98152	.926
.075	.01474 21875	.01627 60254	.00644 79004	.925
.076	.01488 74880	.01646 10206	.00653 60286	.924
.077	.01503 15165	.01664 52102	.00662 41992	.923
.078	.01517 42760	.01682 85957	.00671 24117	.922
.079	.01531 57695	.01701 11785	.00680 06655	.921
.080	.01545 60000	.01719 29600	.00688 89600	.920
.081	.01559 49705	.01737 39416	.00697 72946	.919
.082	.01573 26840	.01755 41248	.00706 56688	.918
.083	.01586 91435	.01773 35110	.00715 40820	.917
.084	.01600 43520	.01791 21015	.00724 25335	.916
.085	.01613 83125	.01808 98979	.00733 10229	.915
.086	.01627 10280	.01826 69015	.00741 95495	.914
.087	.01640 25015	.01844 31137	.00750 81127	.913
.088	.01653 27360	.01861 85359	.00759 67119	.912
.089	.01666 17345	.01879 31697	.00768 53467	.911
.090	.01678 95000	.01896 70162	.00777 40162	.910
.091	.01691 60355	.01914 00771	.00786 27201	.909
.092	.01704 13440	.01931 23537	.00795 14577	.908
.093	.01716 54285	.01948 38474	.00804 02284	.907
.094	.01728 82920	.01965 45595	.00812 90315	.906
.095	.01740 99375	.01982 44916	.00821 78666	.905
.096	.01753 03680	.01999 36451	.00830 67331	.904
.097	.01764 95865	.02016 20212	.00839 56302	.903
.098	.01776 75960	.02032 96215	.00848 45575	.902
.099	.01788 43995	.02049 64473	.00857 35143	.901
.100	.01800 00000	.02066 25000	.00866 25000	.900
	$-A(m)$		$B(m)$	m

TABLE I: COEFFICIENTS OF FOURTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta}$$

$$+ A(m)\left(\frac{\Delta^2}{\Delta}\right)^2 + B(m)\frac{\Delta^4}{\Delta} + \text{5th order terms} + \text{6th order terms}$$

$$+ \text{7th order terms} + \text{8th order terms} + \ldots$$

m	$A(m)$	$B(m)$		m
.100	.01800 00000	.02066 25000	.00866 25000	.900
.101	.01811 44005	.02082 77811	.00875 15141	.899
.102	.01822 76040	.02099 22919	.00884 05559	.898
.103	.01833 96135	.02115 60338	.00892 96248	.897
.104	.01845 04320	.02131 90083	.00901 87203	.896
.105	.01856 00625	.02148 12166	.00910 78416	.895
.106	.01866 85080	.02164 26603	.00919 69883	.894
.107	.01877 57715	.02180 33408	.00928 61598	.893
.108	.01888 18560	.02196 32593	.00937 53553	.892
.109	.01898 67645	.02212 24173	.00946 45743	.891
.110	.01909 05000	.02228 08163	.00955 38162	.890
.111	.01919 30655	.02243 84575	.00964 30805	.889
.112	.01929 44640	.02259 53423	.00973 23663	.888
.113	.01939 46985	.02275 14723	.00982 16733	.887
.114	.01949 37720	.02290 68487	.00991 10007	.886
.115	.01959 16875	.02306 14729	.01000 03479	.885
.116	.01968 84480	.02321 53463	.01008 97143	.884
.117	.01978 40565	.02336 84704	.01017 90994	.883
.118	.01987 85160	.02352 08464	.01026 85024	.882
.119	.01997 18295	.02367 24758	.01035 79228	.881
.120	.02006 40000	.02382 33600	.01044 73600	.880
.121	.02015 50305	.02397 35003	.01053 68133	.879
.122	.02024 49240	.02412 28981	.01062 62821	.878
.123	.02033 36835	.02427 15548	.01071 57658	.877
.124	.02042 13120	.02441 94718	.01080 52638	.876
.125	.02050 78125	.02456 66504	.01089 47754	.875
.126	.02059 31880	.02471 30920	.01098 43000	.874
.127	.02067 74415	.02485 87981	.01107 38371	.873
.128	.02076 05760	.02500 37699	.01116 33859	.872
.129	.02084 25945	.02514 80088	.01125 29458	.871
.130	.02092 35000	.02529 15162	.01134 25162	.870
.131	.02100 32955	.02543 42936	.01143 20966	.869
.132	.02108 19840	.02557 63422	.01152 16862	.868
.133	.02115 95685	.02571 76634	.01161 12844	.867
.134	.02123 60520	.02585 82586	.01170 08906	.866
.135	.02131 14375	.02599 81291	.01179 05041	.865
.136	.02138 57280	.02613 72764	.01188 01244	.864
.137	.02145 89265	.02627 57018	.01196 97508	.863
.138	.02153 10360	.02641 34066	.01205 93826	.862
.139	.02160 20595	.02655 03922	.01214 90192	.861
.140	.02167 20000	.02668 66600	.01223 86600	.860
.141	.02174 08605	.02682 22113	.01232 83043	.859
.142	.02180 86440	.02695 70475	.01241 79515	.858
.143	.02187 53535	.02709 11700	.01250 76010	.857
.144	.02194 09920	.02722 45801	.01259 72521	.856
.145	.02200 55625	.02735 72791	.01268 69041	.855
.146	.02206 90680	.02748 92685	.01277 65565	.854
.147	.02213 15115	.02762 05495	.01286 62085	.853
.148	.02219 28960	.02775 11236	.01295 58596	.852
.149	.02225 32245	.02788 09921	.01304 55091	.851
.150	.02231 25000	.02801 01562	.01313 51562	.850
	$-A(m)$		$B(m)$	m

TABLE I: COEFFICIENTS OF FOURTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta}$$

$$+ A(m)\left(\frac{\Delta^2}{\Delta}\right)^2 + B(m)\frac{\Delta^4}{\Delta} + \text{5th order terms} + \text{6th order terms}$$

$$+ \text{7th order terms} + \text{8th order terms} + \ldots$$

m	$A(m)$	$B(m)$		m
.150	.02231 25000	.02801 01562	.01313 51562	.850
.151	.02237 07255	.02813 86175	.01322 48005	.849
.152	.02242 79040	.02826 63772	.01331 44412	.848
.153	.02248 40385	.02839 34367	.01340 40777	.847
.154	.02253 91320	.02851 97973	.01349 37093	.846
.155	.02259 31875	.02864 54604	.01358 33354	.845
.156	.02264 62080	.02877 04273	.01367 29553	.844
.157	.02269 81965	.02889 46994	.01376 25684	.843
.158	.02274 91560	.02901 82779	.01385 21739	.842
.159	.02279 90895	.02914 11644	.01394 17714	.841
.160	.02284 80000	.02926 33600	.01403 13600	.840
.161	.02289 58905	.02938 48662	.01412 09392	.839
.162	.02294 27640	.02950 56842	.01421 05082	.838
.163	.02298 86235	.02962 58154	.01430 00664	.837
.164	.02303 34720	.02974 52612	.01438 96132	.836
.165	.02307 73125	.02986 40229	.01447 91479	.835
.166	.02312 01480	.02998 21018	.01456 86698	.834
.167	.02316 19815	.03009 94992	.01465 81782	.833
.168	.02320 28160	.03021 62166	.01474 76726	.832
.169	.02324 26545	.03033 22551	.01483 71521	.831
.170	.02328 15000	.03044 76163	.01492 66162	.830
.171	.02331 93555	.03056 23012	.01501 60642	.829
.172	.02335 62240	.03067 63115	.01510 54955	.828
.173	.02339 21085	.03078 96482	.01519 49092	.827
.174	.02342 70120	.03090 23128	.01528 43048	.826
.175	.02346 09375	.03101 43066	.01537 36816	.825
.176	.02349 38880	.03112 56310	.01546 30390	.824
.177	.02352 58665	.03123 62872	.01555 23762	.823
.178	.02355 68760	.03134 62765	.01564 16925	.822
.179	.02358 69195	.03145 56003	.01573 09873	.821
.180	.02361 60000	.03156 42600	.01582 02600	.820
.181	.02364 41205	.03167 22568	.01590 95098	.819
.182	.02367 12840	.03177 95920	.01599 87360	.818
.183	.02369 74935	.03188 62670	.01608 79380	.817
.184	.02372 27520	.03199 22831	.01617 71151	.816
.185	.02374 70625	.03209 76416	.01626 62666	.815
.186	.02377 04280	.03220 23439	.01635 53919	.814
.187	.02379 28515	.03230 63911	.01644 44901	.813
.188	.02381 43360	.03240 97847	.01653 35607	.812
.189	.02383 48845	.03251 25260	.01662 26030	.811
.190	.02385 45000	.03261 46162	.01671 16162	.810
.191	.02387 31855	.03271 60568	.01680 05998	.809
.192	.02389 09440	.03281 68489	.01688 95529	.808
.193	.02390 77785	.03291 69939	.01697 84749	.807
.194	.02392 36920	.03301 64931	.01706 73651	.806
.195	.02393 86875	.03311 53479	.01715 62229	.805
.196	.02395 27680	.03321 35595	.01724 50475	.804
.197	.02396 59365	.03331 11291	.01733 38381	.803
.198	.02397 81960	.03340 80583	01742 25943	.802
.199	.02398 95495	.03350 43481	.01751 13151	.801
.200	.02400 00000	.03360 00000	.01760 00000	.800
	$-A(m)$		$B(m)$	m

TABLE I: COEFFICIENTS OF FOURTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta}$$

$$+ A(m)\left(\frac{\Delta^2}{\Delta}\right)^2 + B(m)\frac{\Delta^4}{\Delta} + \text{5th order terms} + \text{6th order terms}$$

$$+ \text{7th order terms} + \text{8th order terms} + \ldots$$

m	$A(m)$	$B(m)$		m
.200	.02400 00000	.03360 00000	.01760 00000	.800
.201	.02400 95505	.03369 50152	.01768 86482	.799
.202	.02401 82040	.03378 93951	.01777 72591	.798
.203	.02402 59635	.03388 31408	.01786 58318	.797
.204	.02403 28320	.03397 62539	.01795 43659	.796
.205	.02403 88125	.03406 87354	.01804 28604	.795
.206	.02404 39080	.03416 05867	.01813 13147	.794
.207	.02404 81215	.03425 18092	.01821 97282	.793
.208	.02405 14560	.03434 24041	.01830 81001	.792
.209	.02405 39145	.03443 23727	.01839 64297	.791
.210	.02405 55000	.03452 17162	.01848 47162	.790
.211	.02405 62155	.03461 04361	.01857 29591	.789
.212	.02405 60640	.03469 85335	.01866 11575	.788
.213	.02405 50485	.03478 60098	.01874 93108	.787
.214	.02405 31720	.03487 28663	.01883 74183	.786
.215	.02405 04375	.03495 91041	.01892 54791	.785
.216	.02404 68480	.03504 47247	.01901 34927	.784
.217	.02404 24065	.03512 97293	.01910 14583	.783
.218	.02403 71160	.03521 41192	.01918 93752	.782
.219	.02403 09795	.03529 78957	.01927 72427	.781
.220	.02402 40000	.03538 10600	.01936 50600	.780
.221	.02401 61805	.03546 36134	.01945 28264	.779
.222	.02400 75240	.03554 55573	.01954 05413	.778
.223	.02399 80335	.03562 68929	.01962 82039	.777
.224	.02398 77120	.03570 76214	.01971 58134	.776
.225	.02397 65625	.03578 77441	.01980 33691	.775
.226	.02396 45880	.03586 72624	.01989 08704	.774
.227	.02395 17915	.03594 61775	.01997 83165	.773
.228	.02393 81760	.03602 44907	.02006 57067	.772
.229	.02392 37445	.03610 22031	.02015 30401	.771
.230	.02390 85000	.03617 93162	.02024 03162	.770
.231	.02389 24455	.03625 58312	.02032 75342	.769
.232	.02387 55840	.03633 17494	.02041 46934	.768
.233	.02385 79185	.03640 70719	.02050 17929	.767
.234	.02383 94520	.03648 18002	.02058 88322	.766
.235	.02382 01875	.03655 59354	.02067 58104	.765
.236	.02380 01280	.03662 94788	.02076 27268	.764
.237	.02377 92765	.03670 24317	.02084 95807	.763
.238	.02375 76360	.03677 47954	.02093 63714	.762
.239	.02373 52095	.03684 65711	.02102 30981	.761
.240	.02371 20000	.03691 77600	.02110 97600	.760
.241	.02368 80105	.03698 83635	.02119 63565	.759
.242	.02366 32440	.03705 83827	.02128 28867	.758
.243	.02363 77035	.03712 78191	.02136 93501	.757
.244	.02361 13920	.03719 66737	.02145 57457	.756
.245	.02358 43125	.03726 49479	.02154 20729	.755
.246	.02355 64680	.03733 26429	.02162 83309	.754
.247	.02352 78615	.03739 97600	.02171 45190	.753
.248	.02349 84960	.03746 63004	.02180 06364	.752
.249	.02346 83745	.03753 22654	.02188 66824	.751
.250	.02343 75000	.03759 76562	.02197 26562	.750
	$-A(m)$		$B(m)$	m

TABLE I: COEFFICIENTS OF FOURTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta}$$

$$+ A(m)\left(\frac{\Delta^2}{\Delta}\right)^2 + B(m)\frac{\Delta^4}{\Delta} + \text{5th order terms} + \text{6th order terms}$$

$$+ \text{7th order terms} + \text{8th order terms} + \ldots$$

m	$A(m)$	$B(m)$		
.250	.02343 75000	.03759 76562	.02197 26562	.750
.251	.02340 58755	.03766 24742	.02205 85572	.749
.252	.02337 35040	.03772 67204	.02214 43844	.748
.253	.02334 03885	.03779 03962	.02223 01372	.747
.254	.02330 65320	.03785 35020	.02231 58149	.746
.255	.02327 19375	.03791 60416	.02240 14166	.745
.256	.02323 66080	.03797 80137	.02248 69417	.744
.257	.02320 05465	.03803 94203	.02257 23893	.743
.258	.02316 37560	.03810 02627	.02265 77587	.742
.259	.02312 62395	.03816 05422	.02274 30492	.741
.260	.02308 80000	.03822 02600	.02282 82600	.740
.261	.02304 90405	.03827 94173	.02291 33903	.739
.262	.02300 93640	.03833 80154	.02299 84394	.738
.263	.02296 89735	.03839 60555	.02308 34065	.737
.264	.02292 78720	.03845 35388	.02316 82908	.736
.265	.02288 60625	.03851 04666	.02325 30916	.735
.266	.02284 35480	.03856 68402	.02333 78082	.734
.267	.02280 03315	.03862 26607	.02342 24397	.733
.268	.02275 64160	.03867 79294	.02350 69854	.732
.269	.02271 18045	.03873 26475	.02359 14445	.731
.270	.02266 65000	.03878 68162	.02367 58162	.730
.271	.02262 05055	.03884 04369	.02376 00999	.729
.272	.02257 38240	.03889 35107	.02384 42947	.728
.273	.02252 64585	.03894 60388	.02392 83998	.727
.274	.02247 84120	.03899 80224	.02401 24144	.726
.275	.02242 96875	.03904 94629	.02409 63379	.725
.276	.02238 02880	.03910 03614	.02418 01694	.724
.277	.02233 02165	.03915 07191	.02426 39081	.723
.278	.02227 94760	.03920 05373	.02434 75533	.722
.279	.02222 80695	.03924 98172	.02443 11042	.721
.280	.02217 60000	.03929 85600	.02451 45600	.720
.281	.02212 32705	.03934 67669	.02459 79199	.719
.282	.02206 98840	.03939 44392	.02468 11832	.718
.283	.02201 58435	.03944 15781	.02476 43491	.717
.284	.02196 11520	.03948 81847	.02484 74167	.716
.285	.02190 58125	.03953 42604	.02493 03854	.715
.286	.02184 98280	.03957 98063	02501 32543	.714
.287	.02179 32015	.03962 48236	.02509 60226	.713
.288	.02173 59360	.03966 93135	.02517 86895	.712
.289	.02167 80345	.03971 32774	.02526 12544	.711
.290	.02161 95000	.03975 67163	.02534 37162	.710
.291	.02156 03355	.03979 96314	.02542 60744	.709
.292	.02150 05440	.03984 20241	.02550 83281	.708
.293	.02144 01285	.03988 38955	.02559 04765	.707
.294	.02137 90920	.03992 52467	.02567 25187	.706
.295	.02131 74375	.03996 60791	.02575 44541	.705
.296	.02125 51680	.04000 63939	.02583 62819	.704
.297	.02119 22865	.04004 61921	.02591 80011	.703
.298	.02112 87960	.04008 54751	.02599 96111	.702
.299	.02106 46995	.04012 42440	.02608 11110	.701
.300	.02100 00000	.04016 25000	.02616 25000	.700
	$-A(m)$		$B(m)$	m

TABLE I: COEFFICIENTS OF FOURTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta}$$

$$+ A(m)\left(\frac{\Delta^2}{\Delta}\right)^2 + B(m)\frac{\Delta^4}{\Delta} + \text{5th order terms} + \text{6th order terms}$$

$$+ \text{7th order terms} + \text{8th order terms} + \ldots$$

m	$A(m)$	$B(m)$		
.300	.02100 00000	.04016 25000	.02616 25000	.700
.301	.02093 47005	.04020 02444	.02624 37774	.699
.302	.02086 88040	.04023 74783	.02632 49423	.698
.303	.02080 23135	.04027 42029	.02640 59939	.697
.304	.02073 52320	.04031 04195	.02648 69315	.696
.305	.02066 75625	.04034 61291	.02656 77541	.695
.306	.02059 93080	.04038 13331	.02664 84611	.694
.307	.02053 04715	.04041 60327	.02672 90517	.693
.308	.02046 10560	.04045 02289	.02680 95249	.692
.309	.02039 10645	.04048 39230	.02688 98800	.691
.310	.02032 05000	.04051 71162	.02697 01162	.690
.311	.02024 93655	.04054 98098	.02705 02328	.689
.312	.02017 76640	.04058 20047	.02713 02287	.688
.313	.02010 53985	.04061 37024	.02721 01034	.687
.314	.02003 25720	.04064 49039	.02728 98559	.686
.315	.01995 91875	.04067 56104	.02736 94854	.685
.316	.01988 52480	.04070 58231	.02744 89911	.684
.317	.01981 07565	.04073 55433	.02752 83723	.683
.318	.01973 57160	.04076 47720	.02760 76280	.682
.319	.01966 01295	.04079 35105	.02768 67575	.681
.320	.01958 40000	.04082 17600	.02776 57600	.680
.321	.01950 73305	.04084 95216	.02784 46346	.679
.322	.01943 01240	.04087 67965	.02792 33805	.678
.323	.01935 23835	.04090 35859	.02800 19969	.677
.324	.01927 41120	.04092 98910	.02808 04830	.676
.325	.01919 53125	.04095 57129	.02815 88379	.675
.326	.01911 59880	.04098 10528	.02823 70608	.674
.327	.01903 61415	.04100 59120	.02831 51510	.673
.328	.01895 57760	.04103 02915	.02839 31075	.672
.329	.01887 48945	.04105 41925	.02847 09295	.671
.330	.01879 35000	.04107 76162	.02854 86162	.670
.331	.01871 15955	.04110 05639	.02862 61669	.669
.332	.01862 91840	.04112 30366	.02870 35806	.668
.333	.01854 62685	.04114 50355	.02878 08565	.667
.334	.01846 28520	.04116 65618	.02885 79938	.666
.335	.01837 89375	.04118 76166	.02893 49916	.665
.336	.01829 45280	.04120 82012	.02901 18492	.664
.337	.01820 96265	.04122 83167	.02908 85657	.663
.338	.01812 42360	.04124 79642	.02916 51402	.662
.339	.01803 83595	.04126 71449	.02924 15719	.661
.340	.01795 20000	.04128 58600	.02931 78600	.660
.341	.01786 51605	.04130 41106	.02939 40036	.659
.342	.01777 78440	.04132 18979	.02947 00019	.658
.343	.01769 00535	.04133 92231	.02954 58541	.657
.344	.01760 17920	.04135 60873	.02962 15593	.656
.345	.01751 30625	.04137 24916	.02969 71166	.655
.346	.01742 38680	.04138 84373	.02977 25253	.654
.347	.01733 42115	.04140 39254	.02984 77844	.653
.348	.01724 40960	.04141 89572	.02992 28932	.652
.349	.01715 35245	.04143 35338	.02999 78508	.651
.350	.01706 25000	.04144 76562	.03007 26562	.650
	$-A(m)$		$B(m)$	m

TABLE I: COEFFICIENTS OF FOURTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta}$$

$$+ A(m)\left(\frac{\Delta^3}{\Delta}\right)^2 + B(m)\frac{\Delta^4}{\Delta} + \text{5th order terms} + \text{6th order terms}$$

$$+ \text{7th order terms} + \text{8th order terms} + \ldots$$

m	$A(m)$	$B(m)$		
.350	.01706 25000	.04144 76562	.03007 26562	.650
.351	.01697 10255	.04146 13258	.03014 73088	.649
.352	.01687 91040	.04147 45436	.03022 18076	.648
.353	.01678 67385	.04148 73108	.03029 61518	.647
.354	.01669 39320	.04149 96285	.03037 03405	.646
.355	.01660 06875	.04151 14979	.03044 43729	.645
.356	.01650 70080	.04152 29201	.03051 82481	.644
.357	.01641 28965	.04153 38963	.03059 19653	.643
.358	.01631 83560	.04154 44275	.03066 55235	.642
.359	.01622 33895	.04155 45151	.03073 89221	.641
.360	.01612 80000	.04156 41600	.03081 21600	.640
.361	.01603 21905	.04157 33635	.03088 52365	.639
.362	.01593 59640	.04158 21266	.03095 81506	.638
.363	.01583 93235	.04159 04505	.03103 09015	.637
.364	.01574 22720	.04159 83364	.03110 34884	.636
.365	.01564 48125	.04160 57854	.03117 59104	.635
.366	.01554 69480	.04161 27986	.03124 81666	.634
.367	.01544 86815	.04161 93771	.03132 02561	.633
.368	.01535 00160	.04162 55222	.03139 21782	.632
.369	.01525 09545	.04163 12348	.03146 39318	.631
.370	.01515 15000	.04163 65162	.03153 55162	.630
.371	.01505 16555	.04164 13675	.03160 69305	.629
.372	.01495 14240	.04164 57899	.03167 81739	.628
.373	.01485 08085	.04164 97843	.03174 92453	.627
.374	.01474 98120	.04165 33520	.03182 01440	.626
.375	.01464 84375	.04165 64941	.03189 08691	.625
.376	.01454 66880	.04165 92118	.03196 14198	.624
.377	.01444 45665	.04166 15061	.03203 17951	.623
.378	.01434 20760	.04166 33781	.03210 19941	.622
.379	.01423 92195	.04166 48290	.03217 20160	.621
.380	.01413 60000	.04166 58600	.03224 18600	.620
.381	.01403 24205	.04166 64721	.03231 15251	.619
.382	.01392 84840	.04166 66664	.03238 10104	.618
.383	.01382 41935	.04166 64441	.03245 03151	.617
.384	.01371 95520	.04166 58063	.03251 94383	.616
.385	.01361 45625	.04166 47541	.03258 83791	.615
.386	.01350 92280	.04166 32887	.03265 71367	.614
.387	.01340 35515	.04166 14110	.03272 57100	.613
.388	.01329 75360	.04165 91223	.03279 40983	.612
.389	.01319 11845	.04165 64237	.03286 23007	.611
.390	.01308 45000	.04165 33162	.03293 03162	.610
.391	.01297 74855	.04164 98011	.03299 81441	.609
.392	.01287 01440	.04164 58793	.03306 57833	.608
.393	.01276 24785	.04164 15520	.03313 32330	.607
.394	.01265 44920	.04163 68203	.03320 04923	.606
.395	.01254 61875	.04163 16854	.03326 75604	.605
.396	.01243 75680	.04162 61483	.03333 44363	.604
.397	.01232 86365	.04162 02100	.03340 11190	.603
.398	.01221 93960	.04161 38719	.03346 76079	.602
.399	.01210 98495	.04160 71348	.03353 39018	.601
.400	.01200 00000	.04160 00000	.03360 00000	.600
	$-A(m)$		$B(m)$	m

TABLE I: COEFFICIENTS OF FOURTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta}$$

$$+ A(m)\left(\frac{\Delta^2}{\Delta}\right)^2 + B(m)\frac{\Delta^4}{\Delta} + \text{5th order terms} + \text{6th order terms}$$

$$+ \text{7th order terms} + \text{8th order terms} + \ldots$$

m	$A(m)$	$B(m)$		
.400	.01200 00000	.04160 00000	.03360 00000	.600
.401	.01188 98505	.04159 24685	.03366 59015	.599
.402	.01177 94040	.04158 45415	.03373 16055	.598
.403	.01166 86635	.04157 62199	.03379 71109	.597
.404	.01155 76320	.04156 75051	.03386 24171	.596
.405	.01144 63125	.04155 83979	.03392 75229	.595
.406	.01133 47080	.04154 88995	.03399 24275	.594
.407	.01122 28215	.04153 90111	.03405 71301	.593
.408	.01111 06560	.04152 87337	.03412 16297	.592
.409	.01099 82145	.04151 80684	.03418 59254	.591
.410	.01088 55000	.04150 70162	.03425 00162	.590
.411	.01077 25155	.04149 55784	.03431 39014	.589
.412	.01065 92640	.04148 37559	.03437 75799	.588
.413	.01054 57485	.04147 15499	.03444 10509	.587
.414	.01043 19720	.04145 89615	.03450 43135	.586
.415	.01031 79375	.04144 59916	.03456 73666	.585
.416	.01020 36480	.04143 26415	.03463 02095	.584
.417	.01008 91065	.04141 89122	.03469 28412	.583
.418	.00997 43160	.04140 48048	.03475 52608	.582
.419	.00985 92795	.04139 03204	.03481 74674	.581
.420	.00974 40000	.04137 54600	.03487 94600	.580
.421	.00962 84805	.04136 02247	.03494 12377	.579
.422	.00951 27240	.04134 46157	.03500 27997	.578
.423	.00939 67335	.04132 86340	.03506 41450	.577
.424	.00928 05120	.04131 22806	.03512 52726	.576
.425	.00916 40625	.04129 55566	.03518 61816	.575
.426	.00904 73880	.04127 84632	.03524 68712	.574
.427	.00893 04915	.04126 10014	.03530 73404	.573
.428	.00881 33760	.04124 31723	.03536 75883	.572
.429	.00869 60445	.04122 49768	.03542 76138	.571
.430	.00857 85000	.04120 64162	.03548 74162	.570
.431	.00846 07455	.04118 74915	.03554 69945	.569
.432	.00834 27840	.04116 82038	.03560 63478	.568
.433	.00822 46185	.04114 85540	.03566 54750	.567
.434	.00810 62520	.04112 85434	.03572 43754	.566
.435	.00798 76875	.04110 81729	.03578 30479	.565
.436	.00786 89280	.04108 74436	.03584 14916	.564
.437	.00774 99765	.04106 63566	.03589 97056	.563
.438	.00763 08360	.04104 49130	.03595 76890	.562
.439	.00751 15095	.04102 31138	.03601 54408	.561
.440	.00739 20000	.04100 09600	.03607 29600	.560
.441	.00727 23105	.04097 84528	.03613 02458	.559
.442	.00715 24440	.04095 55931	.03618 72971	.558
.443	.00703 24035	.04093 23822	.03624 41132	.557
.444	.00691 21920	.04090 88209	.03630 06929	.556
.445	.00679 18125	.04088 49104	.03635 70354	.555
.446	.00667 12680	.04086 06517	.03641 31397	.554
.447	.00655 05615	.04083 60459	.03646 90049	.553
.448	.00642 96960	.04081 10940	.03652 46300	.552
.449	.00630 86745	.04078 57971	.03658 00141	.551
.450	.00618 75000	.04076 01562	.03663 51562	.550
	$-A(m)$		$B(m)$	m

TABLE I: COEFFICIENTS OF FOURTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta}$$

$$+ A(m)\left(\frac{\Delta^2}{\Delta}\right)^2 + B(m)\frac{\Delta^4}{\Delta} + \text{5th order terms} + \text{6th order terms}$$

$$+ \text{7th order terms} + \text{8th order terms} + \ldots$$

m	$A(m)$	$B(m)$		m
.450	.00618 75000	.04076 01562	.03663 51562	.550
.451	.00606 61755	.04073 41725	.03669 00555	.549
.452	.00594 47040	.04070 78468	.03674 47108	.548
.453	.00582 30885	.04068 11803	.03679 91213	.547
.454	.00570 13320	.04065 41741	.03685 32861	.546
.455	.00557 94375	.04062 68291	.03690 72041	.545
.456	.00545 74080	.04059 91465	.03696 08745	.544
.457	.00533 52465	.04057 11272	.03701 42962	.543
.458	.00521 29560	.04054 27723	.03706 74683	.542
.459	.00509 05395	.04051 40829	.03712 03899	.541
.460	.00496 80000	.04048 50600	.03717 30600	.540
.461	.00484 53405	.04045 57046	.03722 54776	.539
.462	.00472 25640	.04042 60178	.03727 76418	.538
.463	.00459 96735	.04039 60006	.03732 95516	.537
.464	.00447 66720	.04036 56540	.03738 12060	.536
.465	.00435 35625	.04033 49791	.03743 26041	.535
.466	.00423 03480	.04030 39770	.03748 37450	.534
.467	.00410 70315	.04027 26486	.03753 46276	.533
.468	.00398 36160	.04024 09950	.03758 52510	.532
.469	.00386 01045	.04020 90172	.03763 56142	.531
.470	.00373 65000	.04017 67162	.03768 57162	.530
.471	.00361 28055	.04014 40932	.03773 55562	.529
.472	.00348 90240	.04011 11491	.03778 51331	.528
.473	.00336 51585	.04007 78849	.03783 44459	.527
.474	.00324 12120	.04004 43016	.03788 34936	.526
.475	.00311 71875	.04001 04004	.03793 22754	.525
.476	.00299 30880	.03997 61822	.03798 07902	.524
.477	.00286 89165	.03994 16480	.03802 90370	.523
.478	.00274 46760	.03990 67989	.03807 70149	.522
.479	.00262 03695	.03987 16359	.03812 47229	.521
.480	.00249 60000	.03983 61600	.03817 21600	.520
.481	.00237 15705	.03980 03722	.03821 93252	.519
.482	.00224 70840	.03976 42736	.03826 62176	.518
.483	.00212 25435	.03972 78652	.03831 28362	.517
.484	.00199 79520	.03969 11479	.03835 91799	.516
.485	.00187 33125	.03965 41229	.03840 52479	.515
.486	.00174 86280	.03961 67911	.03845 10391	.514
.487	.00162 39015	.03957 91535	.03849 65525	.513
.488	.00149 91360	.03954 12111	.03854 17871	.512
.489	.00137 43345	.03950 29651	.03858 67421	.511
.490	.00124 95000	.03946 44162	.03863 14162	.510
.491	.00112 46355	.03942 55657	.03867 58087	.509
.492	.00099 97440	.03938 64145	.03871 99185	.508
.493	.00087 48285	.03934 69636	.03876 37446	.507
.494	.00074 98920	.03930 72139	.03880 72859	.506
.495	.00062 49375	.03926 71666	.03885 05416	.505
.496	.00049 99680	.03922 68227	.03889 35107	.504
.497	.00037 49865	.03918 61830	.03893 61920	.503
.498	.00024 99960	.03914 52487	.03897 85847	.502
.499	.00012 49995	.03910 40207	.03902 06877	.501
.500	.00000 00000	.03906 25000	.03906 25000	.500
	$-A(m)$		$B(m)$	m

TABLE II: COEFFICIENTS OF FIFTH ORDER TERMS

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^2}{\Delta}$$

$$+ \text{4th order terms} + C(m)\frac{\Delta^2}{\Delta}\frac{\Delta^3}{\Delta} + D(m)\frac{\Delta^5}{\Delta} + \text{6th order terms}$$

$$+ \text{7th order terms} + \text{8th order terms} + \ldots$$

m	$C(m)$	$-D(m)$	m	$C(m)$	$-D(m)$
.000	.00000 00000	.00000 00000	.050	−.01370 57291	.00899 42734
.001	−.00033 20847	.00019 95836	.051	.01392 27991	.00915 43790
.002	.00066 16773	.00039 83357	.052	.01413 77641	.00931 37379
.003	.00098 87859	.00059 62579	.053	.01435 06316	.00947 23517
.004	.00131 34185	.00079 33520	.054	.01456 14090	.00963 02222
.005	−.00163 55830	.00098 96197	.055	−.01477 01039	.00978 73510
.006	.00195 52874	.00118 50629	.056	.01497 67236	.00994 37397
.007	.00227 25396	.00137 96832	.057	.01518 12756	.01009 93900
.008	.00258 73476	.00157 34823	.058	.01538 37674	.01025 43034
.009	.00289 97192	.00176 64621	.059	.01558 42064	.01040 84817
.010	−.00320 96625	.00195 86242	.060	−.01578 26000	.01056 19265
.011	.00351 71852	.00214 99703	.061	.01597 89555	.01071 46393
.012	.00382 22953	.00234 05023	.062	.01617 32805	.01086 66219
.013	.00412 50007	.00253 02217	.063	.01636 55822	.01101 78759
.014	.00442 53093	.00271 91305	.064	.01655 58681	.01116 84028
.015	−.00472 32289	.00290 72302	.065	−.01674 41455	.01131 82044
.016	.00501 87673	.00309 45225	.066	.01693 04218	.01146 72822
.017	.00531 19325	.00328 10093	.067	.01711 47043	.01161 56379
.018	.00560 27322	.00346 66923	.068	.01729 70004	.01176 32730
.019	.00589 11743	.00365 15730	.069	.01747 73173	.01191 01892
.020	−.00617 72666	.00383 56534	.070	−.01765 56625	.01205 63882
.021	.00646 10169	.00401 89350	.071	.01783 20431	.01220 18714
.022	.00674 24330	.00420 14195	.072	.01800 64665	.01234 66406
.023	.00702 15227	.00438 31088	.073	.01817 89400	.01249 06974
.024	.00729 82937	.00456 40044	.074	.01834 94709	.01263 40433
.025	−.00757 27539	.00474 41082	.075	−.01851 80664	.01277 66799
.026	.00784 49109	.00492 34217	.076	.01868 47337	.01291 86089
.027	.00811 47725	.00510 19467	.077	.01884 94802	.01305 98319
.028	.00838 23465	.00527 96849	.078	.01901 23130	.01320 03505
.029	.00864 76406	.00545 66380	.079	.01917 32394	.01334 01662
.030	−.00891 06625	.00563 28077	.080	−.01933 22666	.01347 92806
.031	.00917 14198	.00580 81957	.081	.01948 94018	.01361 76955
.032	.00942 99204	.00598 28036	.082	.01964 46522	.01375 54122
.033	.00968 61718	.00615 66331	.083	.01979 80250	.01389 24325
.034	.00994 01818	.00632 96860	.084	.01994 95273	.01402 87579
.035	−.01019 19580	.00650 19639	.085	−.02009 91664	.01416 43900
.036	.01044 15081	.00667 34685	.086	.02024 69493	.01429 93305
.037	.01068 88397	.00684 42015	.087	.02039 28832	.01443 35808
.038	.01093 39605	.00701 41646	.088	.02053 69753	.01456 71425
.039	.01117 68780	.00718 33593	.089	.02067 92327	.01470 00173
.040	−.01141 76000	.00735 17875	.090	−.02081 96625	.01483 22067
.041	.01165 61339	.00751 94508	.091	.02095 82717	.01496 37123
.042	.01189 24874	.00768 63508	.092	.02109 50676	.01509 45356
.043	.01212 66681	.00785 24892	.093	.02123 00571	.01522 46783
.044	.01235 86836	.00801 78677	.094	.02136 32474	.01535 41419
.045	−.01258 85414	.00818 24879	.095	−.02149 46455	.01548 29280
.046	.01281 62490	.00834 63516	.096	.02162 42585	.01561 10381
.047	.01304 18141	.00850 94603	.097	.02175 20934	.01573 84737
.048	.01326 52441	.00867 18158	.098	.02187 81573	.01586 52366
.049	.01348 65466	.00883 34196	.099	.02200 24571	.01599 13282
.050	−.01370 57291	.00899 42734	.100	−.02212 50000	.01611 67500

TABLE II: COEFFICIENTS OF FIFTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta}$$

$$+ \text{4th order terms} + C(m)\frac{\Delta^2}{\Delta}\frac{\Delta^3}{\Delta} + D(m)\frac{\Delta^5}{\Delta} + \text{6th order terms}$$

$$+ \text{7th order terms} + \text{8th order terms} + \ldots$$

m	$C(m)$	$-D(m)$	m	$C(m)$	$-D(m)$
.100	−.02212 50000	.01611 67500	.150	−.02616 40625	.02156 78203
.101	.02224 57928	.01624 15037	.151	.02620 60656	.02166 11078
.102	.02236 48426	.01636 55907	.152	.02624 66628	.02175 38039
.103	.02248 21564	.01648 90127	.153	.02628 58604	.02184 59102
.104	.02259 77412	.01661 17712	.154	.02632 36650	.02193 74281
.105	−.02271 16039	.01673 38678	.155	−.02636 00830	.02202 83590
.106	.02282 37514	.01685 53039	.156	.02639 51209	.02211 87045
.107	.02293 41908	.01697 60811	.157	.02642 87851	.02220 84659
.108	.02304 29289	.01709 62010	.158	.02646 10821	.02229 76448
.109	.02314 99727	.01721 56652	.159	.02649 20182	.02238 62425
.110	−.02325 53291	.01733 44750	.160	−.02652 16000	.02247 42605
.111	.02335 90050	.01745 26322	.161	.02654 98337	.02256 17002
.112	.02346 10073	.01757 01382	.162	.02657 67258	.02264 85632
.113	.02356 13429	.01768 69945	.163	.02660 22827	.02273 48508
.114	.02366 00186	.01780 32028	.164	.02662 65108	.02282 05644
.115	−.02375 70414	.01791 87644	.165	−.02664 94164	.02290 57056
.116	.02385 24180	.01803 36810	.166	.02667 10058	.02299 02756
.117	.02394 61553	.01814 79541	.167	.02669 12855	.02307 42761
.118	.02403 82602	.01826 15852	.168	.02671 02617	.02315 77084
.119	.02412 87395	.01837 45757	.169	.02672 79408	.02324 05739
.120	−.02421 76000	.01848 69274	.170	−.02674 43291	.02332 28740
.121	.02430 48484	.01859 86415	.171	.02675 94329	.02340 46103
.122	.02439 04917	.01870 97198	.172	.02677 32585	.02348 57841
.123	.02447 45365	.01882 01636	.173	.02678 58122	.02356 63967
.124	.02455 69897	.01892 99745	.174	.02679 71002	.02364 64498
.125	−.02463 78580	.01903 91541	.175	−.02680 71289	.02372 59446
.126	.02471 71482	.01914 77037	.176	.02681 59044	.02380 48826
.127	.02479 48670	.01925 56250	.177	.02682 34330	.02388 32652
.128	.02487 10212	.01936 29194	.178	.02682 97210	.02396 10938
.129	.02494 56174	.01946 95884	.179	.02683 47746	.02403 83698
.130	−.02501 86625	.01957 56336	.180	−.02683 86000	.02411 50946
.131	.02509 01630	.01968 10564	.181	.02684 12033	.02419 12697
.132	.02516 01257	.01978 58583	.182	.02684 25909	.02426 68965
.133	.02522 85573	.01989 00409	.183	.02684 27688	.02434 19763
.134	.02529 54645	.01999 36055	.184	.02684 17433	.02441 65105
.135	−.02536 08539	.02009 65538	.185	−.02683 95205	.02449 05006
.136	.02542 47321	.02019 88872	.186	.02683 61066	.02456 39479
.137	.02548 71059	.02030 06072	.187	.02683 15077	.02463 68539
.138	.02554 79818	.02040 17152	.188	.02682 57300	.02470 92199
.139	.02560 73665	.02050 22129	.189	.02681 87795	.02478 10473
.140	−.02566 52666	.02060 21015	.190	−.02681 06625	.02485 23376
.141	.02572 16887	.02070 13827	.191	.02680 13849	.02492 30921
.142	.02577 66394	.02080 00579	.192	.02679 09529	.02499 33121
.143	.02583 01253	.02089 81286	.193	.02677 93726	.02506 29992
.144	.02588 21529	.02099 55962	.194	.02676 66501	.02513 21546
.145	−.02593 27289	.02109 24622	.195	−.02675 27914	.02520 07797
.146	.02598 18597	.02118 87282	.196	.02673 78025	.02526 88760
.147	.02602 95519	.02128 43955	.197	.02672 16896	.02533 64448
.148	.02607 58121	.02137 94656	.198	.02670 44586	.02540 34875
.149	.02612 06468	.02147 39401	.199	.02668 61156	.02547 00054
.150	−.02616 40625	.02156 78203	.200	−.02666 66666	.02553 60000

TABLE II: COEFFICIENTS OF FIFTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta}$$

$$+ \text{4th order terms} + C(m)\frac{\Delta^2}{\Delta}\frac{\Delta^3}{\Delta} + D(m)\frac{\Delta^5}{\Delta} + \text{6th order terms}$$

$$+ \text{7th order terms} + \text{8th order terms} + \ldots$$

m	$C(m)$	$-D(m)$	m	$C(m)$	$-D(m)$
.200	−.02666 66666	.02553 60000	.250	−.02441 40625	.02819 82422
.201	.02664 61176	.02560 14726	.251	.02434 59488	.02823 93231
.202	.02662 44746	.02566 64245	.252	.02427 70281	.02827 99496
.203	.02660 17436	.02573 08572	.253	.02420 73059	.02832 01229
.204	.02657 79305	.02579 47719	.254	.02413 67877	.02835 98444
.205	−.02655 30414	.02585 81702	.255	−.02406 54789	.02839 91152
.206	.02652 70821	.02592 10532	.256	.02399 33849	.02843 79367
.207	.02650 00586	.02598 34225	.257	.02392 05113	.02847 63100
.208	.02647 19769	.02604 52793	.258	.02384 68634	.02851 42366
.209	.02644 28429	.02610 66250	.259	.02377 24467	.02855 17177
.210	−.02641 26625	.02616 74609	.260	−.02369 72666	.02858 87545
.211	.02638 14415	.02622 77885	.261	.02362 13285	.02862 53483
.212	.02634 91860	.02628 76090	.262	.02354 46378	.02866 15003
.213	.02631 59017	.02634 69238	.263	.02346 71999	.02869 72119
.214	.02628 15946	.02640 57343	.264	.02338 90201	.02873 24842
.215	−.02624 62705	.02646 40418	.265	−.02331 01039	.02876 73186
.216	.02620 99353	.02652 18477	.266	.02323 04565	.02880 17163
.217	.02617 25948	.02657 91532	.267	.02315 00833	.02883 56785
.218	.02613 42549	.02663 59598	.268	.02306 89897	.02886 92065
.219	.02609 49213	.02669 22687	.269	.02298 71810	.02890 23016
.220	−.02605 46000	.02674 80814	.270	−.02290 46625	.02893 49649
.221	.02601 32966	.02680 33990	.271	.02282 14394	.02896 71978
.222	.02597 10170	.02685 82231	.272	.02273 75172	.02899 90015
.223	.02592 77670	.02691 25549	.273	.02265 29010	.02903 03773
.224	.02588 35524	.02696 63957	.274	.02256 75962	.02906 13263
.225	−.02583 83789	.02701 97468	.275	−.02248 16080	.02909 18499
.226	.02579 22522	.02707 26097	.276	.02239 49417	.02912 19492
.227	.02574 51782	.02712 49855	.277	.02230 76025	.02915 16254
.228	.02569 71625	.02717 68758	.278	.02221 95957	.02918 08800
.229	.02564 82109	.02722 82816	.279	.02213 09264	.02920 97140
.230	−.02559 83291	.02727 92045	.280	−.02204 16000	.02923 81286
.231	.02554 75228	.02732 96456	.281	.02195 16215	.02926 61252
.232	.02549 57977	.02737 96063	.282	.02186 09962	.02929 37050
.233	.02544 31595	.02742 90880	.283	.02176 97293	.02932 08691
.234	.02538 96138	.02747 80919	.284	.02167 78260	.02934 76189
.235	−.02533 51664	.02752 66193	.285	−.02158 52914	.02937 39555
.236	.02527 98228	.02757 46717	.286	.02149 21306	.02939 98801
.237	.02522 35887	.02762 22501	.287	.02139 83489	.02942 53940
.238	.02516 64698	.02766 93560	.288	.02130 39513	.02945 04984
.239	.02510 84717	.02771 59907	.289	.02120 89430	.02947 51945
.240	−.02504 96000	.02776 21555	.290	−.02111 33291	.02949 94835
.241	.02498 98602	.02780 78517	.291	.02101 71148	.02952 33666
.242	.02492 92581	.02785 30805	.292	.02092 03049	.02954 68451
.243	.02486 77991	.02789 78432	.293	.02082 29048	.02956 99201
.244	.02480 54889	.02794 21413	.294	.02072 49194	.02959 25929
.245	−.02474 23330	.02798 59759	.295	−.02062 63539	.02961 48646
.246	.02467 83370	.02802 93483	.296	.02052 72132	.02963 67366
.247	.02461 35064	.02807 22599	.297	.02042 75024	.02965 82099
.248	.02454 78468	.02811 47118	.298	.02032 72266	.02967 92857
.249	.02448 13636	.02815 67055	.299	.02022 63908	.02969 99654
.250	−.02441 40625	.02819 82422	.300	−.02012 50000	.02972 02500

TABLE II: COEFFICIENTS OF FIFTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta}$$

$$+ \text{4th order terms} + C(m)\frac{\Delta^3}{\Delta}\frac{\Delta^3}{\Delta} + D(m)\frac{\Delta^5}{\Delta} + \text{6th order terms}$$

$$+ \text{7th order terms} + \text{8th order terms} + \ldots$$

m	$C(m)$	$-D(m)$	m	$C(m)$	$-D(m)$
.300	−.02012 50000	.02972 02500	.350	−.01445 57291	.03025 67891
.301	.02002 30591	.02974 01408	.351	.01433 24486	.03025 84756
.302	.01992 05733	.02975 96389	.352	.01420 88601	.03025 98270
.303	.01981 75474	.02977 87456	.353	.01408 49681	.03026 08445
.304	.01971 39865	.02979 74621	.354	.01396 07770	.03026 15291
.305	−.01960 98955	.02981 57894	.355	−.01383 62914	.03026 18820
.306	.01950 52794	.02983 37289	.356	.01371 15156	.03026 19042
.307	.01940 01431	.02985 12817	.357	.01358 64541	.03026 15968
.308	.01929 44916	.02986 84490	.358	.01346 11114	.03026 09610
.309	.01918 83297	.02988 52320	.359	.01333 54919	.03025 99979
.310	−.01908 16625	.02990 16318	.360	−.01320 96000	.03025 87085
.311	.01897 44947	.02991 76496	.361	.01308 34400	.03025 70939
.312	.01886 68313	.02993 32867	.362	.01295 70165	.03025 51553
.313	.01875 86772	.02994 85441	.363	.01283 03337	.03025 28937
.314	.01865 00373	.02996 34231	.364	.01270 33961	.03025 03102
.315	−.01854 09164	.02997 79249	.365	−.01257 62080	.03024 74060
.316	.01843 13193	.02999 20505	.366	.01244 87738	.03024 41820
.317	.01832 12510	.03000 58012	.367	.01232 10978	.03024 06394
.318	.01821 07162	.03001 91781	.368	.01219 31844	.03023 67793
.319	.01809 97198	.03003 21825	.369	.01206 50378	.03023 26027
.320	−.01798 82666	.03004 48154	.370	−.01193 66625	.03022 81108
.321	.01787 63614	.03005 70780	.371	.01180 80626	.03022 33046
.322	.01776 40090	.03006 89715	.372	.01167 92425	.03021 81851
.323	.01765 12142	.03008 04971	.373	.01155 02065	.03021 27535
.324	.01753 79817	.03009 16558	.374	.01142 09589	.03020 70109
.325	−.01742 43164	.03010 24490	.375	−.01129 15039	.03020 09583
.326	.01731 02229	.03011 28776	.376	.01116 18457	.03019 45967
.327	.01719 57060	.03012 29429	.377	.01103 19887	.03018 79273
.328	.01708 07705	.03013 26460	.378	.01090 19370	.03018 09511
.329	.01696 54211	.03014 19881	.379	.01077 16949	.03017 36692
.330	−.01684 96625	.03015 09703	.380	−.01064 12666	.03016 60826
.331	.01673 34993	.03015 95938	.381	.01051 06563	.03015 81925
.332	.01661 69364	.03016 78596	.382	.01037 98682	.03014 99998
.333	.01649 99783	.03017 57690	.383	.01024 89065	.03014 15057
.334	.01638 26298	.03018 33231	.384	.01011 77753	.03013 27111
.335	−.01626 48955	.03019 05230	.385	−.00998 64789	.03012 36172
.336	.01614 67801	.03019 73699	.386	.00985 50213	.03011 42250
.337	.01602 82882	.03020 38648	.387	.00972 34067	.03010 45356
.338	.01590 94245	.03021 00090	.388	.00959 16393	.03009 45500
.339	.01579 01935	.03021 58035	.389	.00945 97232	.03008 42692
.340	−.01567 06000	.03022 12495	.390	−.00932 76625	.03007 36943
.341	.01555 06484	.03022 63482	.391	.00919 54612	.03006 28264
.342	.01543 03434	.03023 11005	.392	.00906 31236	.03005 16665
.343	.01530 96896	.03023 55078	.393	.00893 06536	.03004 02156
.344	.01518 86916	.03023 95710	.394	.00879 80554	.03002 84748
.345	−.01506 73539	.03024 32914	.395	−.00866 53330	.03001 64452
.346	.01494 56810	.03024 66700	.396	.00853 24905	.03000 41277
.347	.01482 36776	.03024 97079	.397	.00839 95319	.02999 15234
.348	.01470 13481	.03025 24064	.398	.00826 64613	.02997 86333
.349	.01457 86971	.03025 47664	.399	.00813 32826	.02996 54585
.350	−.01445 57291	.03025 67891	.400	−.00800 00000	.02995 20000

TABLE II: COEFFICIENTS OF FIFTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta}$$

$$+ \text{4th order terms} + C(m)\frac{\Delta^3}{\Delta}\frac{\Delta^3}{\Delta} + D(m)\frac{\Delta^5}{\Delta} + \text{6th order terms}$$

$$+ \text{7th order terms} + \text{8th order terms} + \ldots$$

m	$C(m)$	$-D(m)$	m	$C(m)$	$-D(m)$
.400	−.00800 00000	.02995 20000	.450	−.00128 90625	.02893 97109
.401	.00786 66173	.02993 82588	.451	.00115 55651	.02891 31156
.402	.00773 31386	.02992 42360	.452	.00102 21588	.02888 62881
.403	.00759 95679	.02990 99326	.453	.00088 88469	.02885 92293
.404	.00746 59092	.02989 53496	.454	.00075 56330	.02883 19403
.405	−.00733 21664	.02988 04881	.455	−.00062 25205	.02880 44219
.406	.00719 83434	.02986 53490	.456	.00048 95129	.02877 66750
.407	.00706 44443	.02984 99334	.457	.00035 66136	.02874 87007
.408	.00693 04729	.02983 42423	.458	.00022 38261	.02872 04999
.409	.00679 64332	.02981 82767	.459	−.00009 11537	.02869 20735
.410	−.00666 23291	.02980 20377	.460	+.00004 14000	.02866 34225
.411	.00652 81645	.02978 55262	.461	.00017 38317	.02863 45477
.412	.00639 39433	.02976 87433	.462	.00030 61381	.02860 54502
.413	.00625 96694	.02975 16899	.463	.00043 83157	.02857 61308
.414	.00612 53466	.02973 43672	.464	.00057 03611	.02854 65905
.415	−.00599 09789	.02971 67760	.465	.00070 22710	.02851 68303
.416	.00585 65700	.02969 89175	.466	.00083 40421	.02848 68509
.417	.00572 21238	.02968 07925	.467	.00096 56709	.02845 66535
.418	.00558 76442	.02966 24022	.468	.00109 71542	.02842 62389
.419	.00545 31350	.02964 37475	.469	.00122 84886	.02839 56079
.420	−.00531 86000	.02962 48294	.470	.00135 96708	.02836 47617
.421	.00518 40429	.02960 56489	.471	.00149 06975	.02833 37010
.422	.00504 94677	.02958 62070	.472	.00162 15654	.02830 24268
.423	.00491 48780	.02956 65047	.473	.00175 22712	.02827 09400
.424	.00478 02777	.02954 65431	.474	.00188 28117	.02823 92415
.425	−.00464 56705	.02952 63230	.475	.00201 31835	.02820 73323
.426	.00451 10602	.02950 58455	.476	.00214 33835	.02817 52132
.427	.00437 64505	.02948 51116	.477	.00227 34084	.02814 28852
.428	.00424 18452	.02946 41223	.478	.00240 32549	.02811 03491
.429	.00410 72479	.02944 28785	.479	.00253 29198	.02807 76060
.430	−.00397 26625	.02942 13812	.480	.00266 24000	.02804 46566
.431	.00383 80925	.02939 96315	.481	.00279 16921	.02801 15020
.432	.00370 35417	.02937 76302	.482	.00292 07930	.02797 81429
.433	.00356 90138	.02935 53784	.483	.00304 96996	.02794 45804
.434	.00343 45125	.02933 28771	.484	.00317 84086	.02791 08152
.435	−.00330 00414	.02931 01273	.485	.00330 69169	.02787 68484
.436	.00316 56041	.02928 71298	.486	.00343 52213	.02784 26808
.437	.00303 12044	.02926 38857	.487	.00356 33187	.02780 83132
.438	.00289 68458	.02924 03960	.488	.00369 12059	.02777 37467
.439	.00276 25320	.02921 66616	.489	.00381 88799	.02773 89821
.440	−.00262 82666	.02919 26835	.490	.00394 63375	.02770 40202
.441	.00249 40532	.02916 84627	.491	.00407 35755	.02766 88620
.442	.00235 98954	.02914 40001	.492	.00420 05910	.02763 35084
.443	.00222 57968	.02911 92967	.493	.00432 73808	.02759 79602
.444	.00209 17609	.02909 43534	.494	.00445 39418	.02756 22184
.445	−.00195 77914	.02906 91713	.495	.00458 02710	.02752 62838
.446	.00182 38917	.02904 37512	.496	.00470 63654	.02749 01573
.447	.00169 00654	.02901 80942	.497	.00483 22218	.02745 38398
.448	.00155 63161	.02899 22012	.498	.00495 78373	.02741 73322
.449	.00142 26473	.02896 60731	.499	.00508 32088	.02738 06353
.450	−.00128 90625	.02893 97109	.500	.00520 83333	.02734 37500

TABLE II: COEFFICIENTS OF FIFTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta}$$

$$+ \text{4th order terms} + C(m)\frac{\Delta^3}{\Delta}\frac{\Delta^3}{\Delta} + D(m)\frac{\Delta^5}{\Delta} + \text{6th order terms}$$

$$+ \text{7th order terms} + \text{8th order terms} + \ldots$$

m	$C(m)$	$-D(m)$	m	$C(m)$	$-D(m)$
.500	.00520 83333	.02734 37500	.550	.01108 59375	.02527 82578
.501	.00533 32078	.02730 66772	.551	.01119 47016	.02523 28937
.502	.00545 78293	.02726 94178	.552	.01130 30758	.02518 73849
.503	.00558 21948	.02723 19727	.553	.01141 10575	.02514 17320
.504	.00570 63014	.02719 43427	.554	.01151 86442	.02509 59359
.505	.00583 01460	.02715 65286	.555	.01162 58335	.02504 99974
.506	.00595 37258	.02711 85314	.556	.01173 26230	.02500 39173
.507	.00607 70378	.02708 03520	.557	.01183 90101	.02495 76963
.508	.00620 00790	.02704 19911	.558	.01194 49925	.02491 13354
.509	.00632 28465	.02700 34497	.559	.01205 05677	.02486 48351
.510	.00644 53375	.02696 47285	.560	.01215 57333	.02481 81965
.511	.00656 75489	.02692 58286	.561	.01226 04869	.02477 14202
.512	.00668 94779	.02688 67507	.562	.01236 48261	.02472 45069
.513	.00681 11217	.02684 74957	.563	.01246 87485	.02467 74576
.514	.00693 24773	.02680 80644	.564	.01257 22518	.02463 02730
.515	.00705 35419	.02676 84578	.565	.01267 53335	.02458 29539
.516	.00717 43126	.02672 86766	.566	.01277 79914	.02453 55010
.517	.00729 47866	.02668 87217	.567	.01288 02231	.02448 79152
.518	.00741 49610	.02664 85940	.568	.01298 20262	.02444 01971
.519	.00753 48331	.02660 82942	.569	.01308 33984	.02439 23477
.520	.00765 44000	.02656 78234	.570	.01318 43375	.02434 43675
.521	.00777 36588	.02652 71822	.571	.01328 48410	.02429 62576
.522	.00789 26069	.02648 63716	.572	.01338 49067	.02424 80185
.523	.00801 12414	.02644 53923	.573	.01348 45324	.02419 96511
.524	.00812 95595	.02640 42453	.574	.01358 37157	.02415 11562
.525	.00824 75585	.02636 29314	.575	.01368 24544	.02410 25344
.526	.00836 52357	.02632 14514	.576	.01378 07462	.02405 37867
.527	.00848 25882	.02627 98061	.577	.01387 85889	.02400 49136
.528	.00859 96134	.02623 79964	.578	.01397 59802	.02395 59161
.529	.00871 63085	.02619 60231	.579	.01407 29180	.02390 67949
.530	.00883 26708	.02615 38871	.580	.01416 94000	.02385 75506
.531	.00894 86976	.02611 15891	.581	.01426 54239	.02380 81842
.532	.00906 43862	.02606 91301	.582	.01436 09877	.02375 86963
.533	.00917 97339	.02602 65108	.583	.01445 60891	.02370 90877
.534	.00929 47381	.02598 37320	.584	.01455 07259	.02365 93592
.535	.00940 93960	.02594 07947	.585	.01464 48960	.02360 95114
.536	.00952 37051	.02589 76995	.586	.01473 85973	.02355 95452
.537	.00963 76627	.02585 44474	.587	.01483 18275	.02350 94614
.538	.00975 12661	.02581 10392	.588	.01492 45846	.02345 92605
.539	.00986 45127	.02576 74756	.589	.01501 68664	.02340 89435
.540	.00997 74000	.02572 37575	.590	.01510 86708	.02335 85111
.541	.01008 99252	.02567 98858	.591	.01519 99957	.02330 79639
.542	.01020 20858	.02563 58611	.592	.01529 08390	.02325 73028
.543	.01031 38793	.02559 16844	.593	.01538 11986	.02320 65285
.544	.01042 53030	.02554 73565	.594	.01547 10725	.02315 56416
.545	.01053 63544	.02550 28781	.595	.01556 04585	.02310 46431
.546	.01064 70309	.02545 82500	.596	.01564 93547	.02305 35335
.547	.01075 73300	.02541 34732	.597	.01573 77590	.02300 23137
.548	.01086 72491	.02536 85483	.598	.01582 56693	.02295 09844
.549	.01097 67858	.02532 34763	.599	.01591 30836	.02289 95462
.550	.01108 59375	.02527 82578	.600	.01600 00000	.02284 80000

TABLE II: COEFFICIENTS OF FIFTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta}$$

$$+ \text{4th order terms} + C(m)\frac{\Delta^2}{\Delta}\frac{\Delta^3}{\Delta} + D(m)\frac{\Delta^5}{\Delta} + \text{6th order terms}$$

$$+ \text{7th order terms} + \text{8th order terms} + \cdots$$

m	$C(m)$	$-D(m)$	m	$C(m)$	$-D(m)$
.600	.01600 00000	.02284 80000	.650	.01966 92708	.02014 86797
.601	.01608 64163	.02279 63465	.651	.01972 83518	.02009 25604
.602	.01617 23306	.02274 45863	.652	.01978 68438	.02003 63693
.603	.01625 77410	.02269 27203	.653	.01984 47453	.01998 01069
.604	.01634 26454	.02264 07491	.654	.01990 20549	.01992 37739
.605	.01642 70419	.02258 86735	.655	.01995 87710	.01986 73710
.606	.01651 09285	.02253 64942	.656	.02001 48923	.01981 08989
.607	.01659 43033	.02248 42119	.657	.02007 04173	.01975 43581
.608	.01667 71643	.02243 18274	.658	.02012 53445	.01969 77493
.609	.01675 95097	.02237 93413	.659	.02017 96725	.01964 10732
.610	.01684 13375	.02232 67544	.660	.02023 34000	.01958 43305
.611	.01692 26457	.02227 40674	.661	.02028 65254	.01952 75217
.612	.01700 34326	.02222 12810	.662	.02033 90474	.01947 06476
.613	.01708 36962	.02216 83960	.663	.02039 09647	.01941 37087
.614	.01716 34346	.02211 54130	.664	.02044 22758	.01935 67058
.615	.01724 26460	.02206 23327	.665	.02049 29794	.01929 96394
.616	.01732 13286	.02200 91559	.666	.02054 30741	.01924 25103
.617	.01739 94804	.02195 58832	.667	.02059 25586	.01918 53189
.618	.01747 70997	.02190 25155	.668	.02064 14315	.01912 80661
.619	.01755 41846	.02184 90533	.669	.02068 96916	.01907 07524
.620	.01763 07333	.02179 54974	.670	.02073 73375	.01901 33784
.621	.01770 67440	.02174 18484	.671	.02078 43678	.01895 59449
.622	.01778 22149	.02168 81072	.672	.02083 07814	.01889 84523
.623	.01785 71442	.02163 42744	.673	.02087 65769	.01884 09014
.624	.01793 15302	.02158 03506	.674	.02092 17530	.01878 32929
.625	.01800 53710	.02152 63367	.675	.02096 63085	.01872 56272
.626	.01807 86650	.02147 22332	.676	.02101 02422	.01866 79051
.627	.01815 14104	.02141 80409	.677	.02105 35527	.01861 01271
.628	.01822 36054	.02136 37604	.678	.02109 62389	.01855 22940
.629	.01829 52483	.02130 93926	.679	.02113 82995	.01849 44063
.630	.01836 63375	.02125 49380	.680	.02117 97333	.01843 64646
.631	.01843 68711	.02120 03973	.681	.02122 05391	.01837 84697
.632	.01850 68475	.02114 57712	.682	.02126 07157	.01832 04220
.633	.01857 62651	.02109 10605	.683	.02130 02619	.01826 23222
.634	.01864 51221	.02103 62657	.684	.02133 91766	.01820 41709
.635	.01871 34169	.02098 13877	.685	.02137 74585	.01814 59688
.636	.01878 11478	.02092 64270	.686	.02141 51066	.01808 77165
.637	.01884 83132	.02087 13844	.687	.02145 21197	.01802 94145
.638	.01891 49114	.02081 62605	.688	.02148 84966	.01797 10635
.639	.01898 09409	.02076 10559	.689	.02152 42362	.01791 26641
.640	.01904 64000	.02070 57715	.690	.02155 93375	.01785 42170
.641	.01911 12870	.02065 04079	.691	.02159 37992	.01779 57226
.642	.01917 56005	.02059 49656	.692	.02162 76203	.01773 71817
.643	.01923 93388	.02053 94455	.693	.02166 07998	.01767 85948
.644	.01930 25003	.02048 38481	.694	.02169 33365	.01761 99625
.645	.01936 50835	.02042 81742	.695	.02172 52294	.01756 12855
.646	.01942 70869	.02037 24244	.696	.02175 64774	.01750 25643
.647	.01948 85088	.02031 65994	.697	.02178 70795	.01744 37996
.648	.01954 93478	.02026 06998	.698	.02181 70346	.01738 49919
.649	.01960 96023	.02020 47264	.699	.02184 63418	.01732 61418
.650	.01966 92708	.02014 86797	.700	.02187 50000	.01726 72500

TABLE II: COEFFICIENTS OF FIFTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta}$$

$$+ \text{4th order terms} + C(m)\frac{\Delta^2}{\Delta}\frac{\Delta^3}{\Delta} + D(m)\frac{\Delta^5}{\Delta} + \text{6th order terms}$$

$$+ \text{7th order terms} + \text{8th order terms} + \cdots$$

m	$C(m)$	$-D(m)$	m	$C(m)$	$-D(m)$
.700	.02187 50000	.01726 72500	.750	.02246 09375	.01428 22266
.701	.02190 30081	.01720 83170	.751	.02245 53853	.01422 19662
.702	.02193 03653	.01714 93435	.752	.02244 91451	.01416 16934
.703	.02195 70705	.01709 03299	.753	.02244 22165	.01410 14086
.704	.02198 31227	.01703 12770	.754	.02243 45989	.01404 11124
.705	.02200 85210	.01697 21853	.755	.02242 62919	.01398 08053
.706	.02203 32645	.01691 30553	.756	.02241 72950	.01392 04878
.707	.02205 73521	.01685 38878	.757	.02240 76078	.01386 01605
.708	.02208 07830	.01679 46832	.758	.02239 72298	.01379 98238
.709	.02210 35562	.01673 54422	.759	.02238 61607	.01373 94783
.710	.02212 56708	.01667 61653	.760	.02237 44000	.01367 91245
.711	.02214 71259	.01661 68531	.761	.02236 19472	.01361 87629
.712	.02216 79206	.01655 75062	.762	.02234 88021	.01355 83941
.713	.02218 80540	.01649 81252	.763	.02233 49642	.01349 80186
.714	.02220 75253	.01643 87107	.764	.02232 04331	.01343 76368
.715	.02222 63335	.01637 92632	.765	.02230 52085	.01337 72493
.716	.02224 44779	.01631 97833	.766	.02228 92901	.01331 68567
.717	.02226 19576	.01626 02716	.767	.02227 26774	.01325 64593
.718	.02227 87717	.01620 07287	.768	.02225 53702	.01319 60578
.719	.02229 49194	.01614 11551	.769	.02223 73681	.01313 56526
.720	.02231 04000	.01608 15514	.770	.02221 86708	.01307 52443
.721	.02232 52125	.01602 19181	.771	.02219 92780	.01301 48333
.722	.02233 93562	.01596 22559	.772	.02217 91894	.01295 44202
.723	.02235 28304	.01590 25654	.773	.02215 84047	.01289 40055
.724	.02236 56342	.01584 28470	.774	.02213 69237	.01283 35896
.725	.02237 77669	.01578 31013	.775	.02211 47460	.01277 31731
.726	.02238 92277	.01572 33290	.776	.02209 18715	.01271 27565
.727	.02240 00159	.01566 35305	.777	.02206 82999	.01265 23402
.728	.02241 01307	.01560 37064	.778	.02204 40309	.01259 19248
.729	.02241 95715	.01554 38574	.779	.02201 90643	.01253 15108
.730	.02242 83375	.01548 39838	.780	.02199 34000	.01247 10986
.731	.02243 64279	.01542 40864	.781	.02196 70376	.01241 06888
.732	.02244 38422	.01536 41656	.782	.02193 99770	.01235 02819
.733	.02245 05796	.01530 42221	.783	.02191 22181	.01228 98783
.734	.02245 66394	.01524 42563	.784	.02188 37606	.01222 94785
.735	.02246 20210	.01518 42688	.785	.02185 46044	.01216 90831
.736	.02246 67238	.01512 42602	.786	.02182 47493	.01210 86925
.737	.02247 07470	.01506 42311	.787	.02179 41952	.01204 83071
.738	.02247 40901	.01500 41819	.788	.02176 29419	.01198 79276
.739	.02247 67524	.01494 41132	.789	.02173 09894	.01192 75543
.740	.02247 87333	.01488 40255	.790	.02169 83375	.01186 71878
.741	.02248 00322	.01482 39195	.791	.02166 49860	.01180 68286
.742	.02248 06485	.01476 37956	.792	.02163 09350	.01174 64770
.743	.02248 05816	.01470 36544	.793	.02159 61843	.01168 61337
.744	.02247 98310	.01464 34964	.794	.02156 07338	.01162 57990
.745	.02247 83960	.01458 33222	.795	.02152 45835	.01156 54735
.746	.02247 62762	.01452 31323	.796	.02148 77334	.01150 51576
.747	.02247 34710	.01446 29273	.797	.02145 01833	.01144 48519
.748	.02246 99798	.01440 27076	.798	.02141 19333	.01138 45567
.749	.02246 58021	.01434 24739	.799	.02137 29833	.01132 42726
.750	.02246 09375	.01428 22266	.800	.02133 33333	.01126 40000

TABLE II: COEFFICIENTS OF FIFTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta}$$

$$+ \text{4th order terms} + C(m)\frac{\Delta^2}{\Delta}\frac{\Delta^3}{\Delta} + D(m)\frac{\Delta^5}{\Delta} + \text{6th order terms}$$

$$+ \text{7th order terms} + \text{8th order terms} + \cdots$$

m	$C(m)$	$-D(m)$	m	$C(m)$	$-D(m)$
.800	.02133 33333	.01126 40000	.850	.01846 09375	.00827 51484
.801	.02129 29833	.01120 37394	.851	.01838 58021	.00821 60616
.802	.02125 19333	.01114 34913	.852	.01830 99798	.00815 70092
.803	.02121 01833	.01108 32561	.853	.01823 34710	.00809 79917
.804	.02116 77334	.01102 30343	.854	.01815 62762	.00803 90094
.805	.02112 45835	.01096 28264	.855	.01807 83960	.00798 00627
.806	.02108 07338	.01090 26329	.856	.01799 98310	.00792 11521
.807	.02103 61843	.01084 24541	.857	.01792 05816	.00786 22780
.808	.02099 09350	.01078 22906	.858	.01784 06485	.00780 34408
.809	.02094 49860	.01072 21428	.859	.01776 00322	.00774 46408
.810	.02089 83375	.01066 20112	.860	.01767 87333	.00768 58785
.811	.02085 09894	.01060 18962	.861	.01759 67524	.00762 71543
.812	.02080 29419	.01054 17983	.862	.01751 40901	.00756 84685
.813	.02075 41952	.01048 17180	.863	.01743 07470	.00750 98216
.814	.02070 47493	.01042 16557	.864	.01734 67238	.00745 12140
.815	.02065 46044	.01036 16119	.865	.01726 20210	.00739 26461
.816	.02060 37606	.01030 15869	.866	.01717 66394	.00733 41182
.817	.02055 22181	.01024 15814	.867	.01709 05796	.00727 56308
.818	.02049 99770	.01018 15956	.868	.01700 38422	.00721 71842
.819	.02044 70376	.01012 16301	.869	.01691 64279	.00715 87789
.820	.02039 34000	.01006 16854	.870	.01682 83375	.00710 04152
.821	.02033 90643	.01000 17618	.871	.01673 95715	.00704 20935
.822	.02028 40309	.00994 18598	.872	.01665 01307	.00698 38142
.823	.02022 82999	.00988 19798	.873	.01656 00159	.00692 55777
.824	.02017 18715	.00982 21224	.874	.01646 92277	.00686 73844
.825	.02011 47460	.00976 22878	.875	.01637 77669	.00680 92346
.826	.02005 69237	.00970 24767	.876	.01628 56342	.00675 11288
.827	.01999 84047	.00964 26894	.877	.01619 28304	.00669 30673
.828	.01993 91894	.00958 29263	.878	.01609 93562	.00663 50505
.829	.01987 92780	.00952 31879	.879	.01600 52125	.00657 70789
.830	.01981 86708	.00946 34747	.880	.01591 04000	.00651 91526
.831	.01975 73681	.00940 37870	.881	.01581 49194	.00646 12723
.832	.01969 53702	.00934 41253	.882	.01571 87717	.00640 34381
.833	.01963 26774	.00928 44901	.883	.01562 19576	.00634 56506
.834	.01956 92901	.00922 48817	.884	.01552 44779	.00628 79100
.835	.01950 52085	.00916 53006	.885	.01542 63335	.00623 02167
.836	.01944 04331	.00910 57472	.886	.01532 75253	.00617 25712
.837	.01937 49642	.00904 62220	.887	.01522 80540	.00611 49738
.838	.01930 88021	.00898 67254	.888	.01512 79206	.00605 74248
.839	.01924 19472	.00892 72577	.889	.01502 71259	.00599 99247
.840	.01917 44000	.00886 78195	.890	.01492 56708	.00594 24737
.841	.01910 61607	.00880 84112	.891	.01482 35562	.00588 50723
.842	.01903 72298	.00874 90331	.892	.01472 07830	.00582 77209
.843	.01896 76078	.00868 96857	.893	.01461 73521	.00577 04197
.844	.01889 72950	.00863 03694	.894	.01451 32645	.00571 31692
.845	.01882 62919	.00857 10846	.895	.01440 85210	.00565 59697
.846	.01875 45989	.00851 18318	.896	.01430 31227	.00559 88215
.847	.01868 22165	.00845 26114	.897	.01419 70705	.00554 17251
.848	.01860 91451	.00839 34237	.898	.01409 03653	.00548 46809
.849	.01853 53853	.00833 42693	.899	.01398 30081	.00542 76890
.850	.01846 09375	.00827 51484	.900	.01387 50000	.00537 07500

TABLE II: COEFFICIENTS OF FIFTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta}$$

$$+ \text{4th order terms} + C(m)\frac{\Delta^3}{\Delta}\frac{\Delta^3}{\Delta} + D(m)\frac{\Delta^5}{\Delta} + \text{6th order terms}$$

$$+ \text{7th order terms} + \text{8th order terms} + \cdots$$

m	$C(m)$	$-D(m)$	m	$C(m)$	$-D(m)$
.900	.01387 50000	.00537 07500	.950	.00766 92708	.00259 86953
.901	.01376 63418	.00531 38641	.951	.00752 96023	.00254 48971
.902	.01365 70346	.00525 70318	.952	.00738 93478	.00249 11687
.903	.01354 70795	.00520 02533	.953	.00724 85088	.00243 75105
.904	.01343 64774	.00514 35291	.954	.00710 70869	.00238 39227
.905	.01332 52294	.00508 68595	.955	.00696 50835	.00233 04057
.906	.01321 33365	.00503 02447	.956	.00682 25003	.00227 69598
.907	.01310 07998	.00497 36853	.957	.00667 93388	.00222 35852
.908	.01298 76203	.00491 71814	.958	.00653 56005	.00217 02823
.909	.01287 37992	.00486 07336	.959	.00639 12870	.00211 70513
.910	.01275 93375	.00480 43420	.960	.00624 64000	.00206 38925
.911	.01264 42362	.00474 80072	.961	.00610 09409	.00201 08062
.912	.01252 84966	.00469 17293	.962	.00595 49114	.00195. 77928
.913	.01241 21197	.00463 55088	.963	.00580 83132	.00190 48525
.914	.01229 51066	.00457 93459	.964	.00566 11478	.00185 19855
.915	.01217 74585	.00452 32411	.965	.00551 34169	.00179 91922
.916	.01205 91766	.00446 71947	.966	.00536 51221	.00174 64729
.917	.01194 02619	.00441 12070	.967	.00521 62651	.00169 38279
.918	.01182 07157	.00435 52783	.968	.00506 68475	.00164 12574
.919	.01170 05391	.00429 94090	.969	.00491 68711	.00158 87617
.920	.01157 97333	.00424 35994	.970	.00476 63375	.00153 63410
.921	.01145 82995	.00418 78498	.971	.00461 52483	.00148 39958
.922	.01133 62389	.00413 21606	.972	.00446 36054	.00143 17262
.923	.01121 35527	.00407 65322	.973	.00431 14104	.00137 95326
.924	.01109 02422	.00402 09648	.974	.00415 86650	.00132 74152
.925	.01096 63085	.00396 54587	.975	.00400 53710	.00127 53743
.926	.01084 17530	.00391 00144	.976	.00385 15302	.00122 34101
.927	.01071 65769	.00385 46321	.977	.00369 71442	.00117 15230
.928	.01059 07814	.00379 93121	.978	.00354 22149	.00111 97132
.929	.01046 43678	.00374 40549	.979	.00338 67440	.00106 79810
.930	.01033 73375	.00368 88606	.980	.00323 07333	.00101 63266
.931	.01020 96916	.00363 37296	.981	.00307 41846	.00096 47504
.932	.01008 14315	.00357 86623	.982	.00291 70997	.00091 32526
.933	.00995 25586	.00352 36590	.983	.00275 94804	.00086 18335
.934	.00982 30741	.00346 87199	.984	.00260 13286	.00081 04933
.935	.00969 29794	.00341 38455	.985	.00244 26460	.00075 92323
.936	.00956 22758	.00335 90360	.986	.00228 34346	.00070 80507
.937	.00943 09647	.00330 42917	.987	.00212 36962	.00065 69489
.938	.00929 90474	.00324 96130	.988	.00196 34326	.00060 59271
.939	.00916 65254	.00319 50002	.989	.00180 26457	.00055 49856
.940	.00903 34000	.00314 04535	.990	.00164 13375	.00050 41246
.941	.00889 96725	.00308 59734	.991	.00147 95097	.00045 33444
.942	.00876 53445	.00303 15600	.992	.00131 71643	.00040 26452
.943	.00863 04173	.00297 72138	.993	.00115 43033	.00035 20273
.944	.00849 48923	.00292 29350	.994	.00099 09285	.00030 14909
.945	.00835 87710	.00286 87239	.995	.00082 70419	.00025 10364
.946	.00822 20549	.00281 45809	.996	.00066 26454	.00020 06640
.947	.00808 47453	.00276 05062	.997	.00049 77410	.00015 03739
.948	.00794 68438	.00270 65002	.998	.00033 23306	.00010 01663
.949	.00780 83518	.00265 25631	.999	.00016 64163	.00005 00416
.950	.00766 92708	.00259 86953	1.000	.00000 00000	.00000 00000

TABLE III: COEFFICIENTS OF SIXTH ORDER TERMS

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta} + \text{4th order terms} + \text{5th order terms} + E(m)\left(\frac{\Delta^5}{\Delta}\right)^5 + F(m)\left(\frac{\Delta^5}{\Delta}\right)^2 + G(m)\frac{\Delta^3}{\Delta}\frac{\Delta^4}{\Delta} + H(m)\frac{\Delta^6}{\Delta} + \text{7th order terms} + \text{8th order terms} + \cdots$$

m	$E(m)$	$F(m)$	$G(m)$	$H(m)$	m
.00	.00000 00000	.00000 00000	.00000 00000	.00000 00000	.00
.01	.00117 62438	.00106 18292	.00239 51594	.00162 89224	.01
.02	.00220 99000	.00202 79336	.00458 62171	.00318 35923	.02
.03	.00310 82438	.00290 24979	.00658 14124	.00466 58424	.03
.04	.00387 84000	.00368 96085	.00838 88128	.00607 74777	.04
.05	.00452 73438	.00439 32552	.01001 63151	.00742 02756	.05
.06	.00506 19000	.00501 73315	.01147 16472	.00869 59861	.06
.07	.00548 87438	.00556 56359	.01276 23695	.00990 63323	.07
.08	.00581 44000	.00604 18731	.01389 58763	.01105 30101	.08
.09	.00604 52438	.00644 96546	.01487 93975	.01213 76892	.09
.10	.00618 75000	.00679 25000	.01572 00000	.01316 20125	.10
.11	.00624 72438	.00707 38379	.01642 45892	.01412 75972	.11
.12	.00623 04000	.00729 70069	.01699 99104	.01503 60343	.12
.13	.00614 27438	.00746 52566	.01745 25505	.01588 88893	.13
.14	.00598 99000	.00758 17485	.01778 89395	.01668 77022	.14
.15	.00577 73438	.00764 95573	.01801 53516	.01743 39881	.15
.16	.00551 04000	.00767 16715	.01813 79072	.01812 92368	.16
.17	.00519 42438	.00765 09946	.01816 25743	.01877 49136	.17
.18	.00483 39000	.00759 03464	.01809 51696	.01937 24594	.18
.19	.00443 42438	.00749 24633	.01794 13606	.01992 32906	.19
.20	.00400 00000	.00736 00000	.01770 66667	.02042 88000	.20
.21	.00353 57438	.00719 55300	.01739 64606	.02089 03563	.21
.22	.00304 59000	.00700 15469	.01701 59704	.02130 93048	.22
.23	.00253 47438	.00678 04654	.01657 02803	.02168 69675	.23
.24	.00200 64000	.00653 46219	.01606 43328	.02202 46434	.24
.25	.00146 48438	.00626 62760	.01550 29297	.02232 36084	.25
.26	.00091 39000	.00597 76115	.01489 07339	.02258 51160	.26
.27	+.00035 72438	.00567 07367	.01423 22707	.02281 03973	.27
.28	−.00020 16000	.00534 76864	.01353 19296	.02300 06612	.28
.29	−.00075 92562	.00501 04221	.01279 39654	.02315 70945	.29
.30	−.00131 25000	.00466 08333	.01202 25000	.02328 08625	.30
.31	−.00185 82562	.00430 07388	.01122 15238	.02337 31089	.31
.32	−.00239 36000	.00393 18869	.01039 48971	.02343 49560	.32
.33	−.00291 57562	.00355 59574	.00954 63518	.02346 75052	.33
.34	−.00342 21000	.00317 45619	.00867 94928	.02347 18371	.34
.35	−.00391 01562	.00278 92448	.00779 77995	.02344 90115	.35
.36	−.00437 76000	.00240 14848	.00690 46272	.02340 00679	.36
.37	−.00482 22562	.00201 26955	.00600 32088	.02332 60255	.37
.38	−.00524 21000	.00162 42264	.00509 66563	.02322 78836	.38
.39	−.00563 52562	.00123 73642	.00418 79619	.02310 66218	.39
.40	−.00600 00000	.00085 33333	.00328 00000	.02296 32000	.40
.41	−.00633 47562	.00047 32975	.00237 55286	.02279 85588	.41
.42	−.00663 81000	+.00009 83603	.00147 71904	.02261 36197	.42
.43	−.00690 87562	−.00027 04338	+.00058 75149	.02240 92853	.43
.44	−.00714 50000	−.00063 20981	−.00029 10805	.02218 64395	.44
.45	−.00734 76562	−.00098 57031	−.00115 62891	.02194 59475	.45
.46	−.00751 41000	−.00133 03752	−.00200 59128	.02168 86563	.46
.47	−.00764 42562	−.00166 52958	−.00283 78614	.02141 53951	.47
.48	−.00773 76000	−.00198 97003	−.00365 01504	.02112 69747	.48
.49	−.00779 37562	−.00230 28771	−.00444 09000	.02082 41885	.49
.50	−.00781 25000	−.00260 41667	−.00520 83333	.02050 78125	.50

TABLE III: COEFFICIENTS OF SIXTH ORDER TERMS—*Continued*

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta} + 4\text{th}$$

$$\text{order terms} + 5\text{th order terms} + E(m)\left(\frac{\Delta^2}{\Delta}\right)^3 + F(m)\left(\frac{\Delta^2}{\Delta}\right)^2 + G(m)\frac{\Delta^2}{\Delta}\frac{\Delta^4}{\Delta} + H(m)\frac{\Delta^6}{\Delta}$$

$$+ 7\text{th order terms} + 8\text{th order terms} + \cdots$$

m	$E(m)$	$F(m)$	$G(m)$	$H(m)$	m
.50	−.00781 25000	−.00260 41667	−.00520 83333	.02050 78125	.50
.51	−.00779 37562	−.00289 29604	−.00595 07750	.02017 86052	.51
.52	−.00773 76000	−.00316 86997	−.00666 66496	.01983 73081	.52
.53	−.00764 42562	−.00343 08751	−.00735 44803	.01948 46459	.53
.54	−.00751 41000	−.00367 90248	−.00801 28872	.01912 13264	.54
.55	−.00734 76562	−.00391 27344	−.00864 05859	.01874 80412	.55
.56	−.00714 56000	−.00413 16352	−.00923 63861	.01836 54654	.56
.57	−.00690 87562	−.00433 54037	−.00979 91899	.01797 42580	.57
.58	−.00663 81000	−.00452 37603	−.01032 79904	.01757 50623	.58
.59	−.00633 47562	−.00469 64683	−.01082 18702	.01716 85056	.59
.60	−.00600 00000	−.00485 33333	−.01128 00000	.01675 52000	.60
.61	−.00563 52562	−.00499 42017	−.01170 16369	.01633 57420	.61
.62	−.00524 21000	−.00511 89597	−.01208 61229	.01591 07131	.62
.63	−.00482 22562	−.00522 75330	−.01243 28838	.01548 06798	.63
.64	−.00437 76000	−.00531 98848	−.01274 14272	.01504 61940	.64
.65	−.00391 01562	−.00539 60156	−.01301 13411	.01460 77928	.65
.66	−.00342 21000	−.00545 59619	−.01324 22928	.01416 59990	.66
.67	−.00291 57562	−.00549 97949	−.01343 40268	.01372 13214	.67
.68	−.00239 36000	−.00552 76203	−.01358 63637	.01327 42545	.68
.69	−.00185 82562	−.00553 95763	−.01369 91988	.01282 52792	.69
.70	−.00131 25000	−.00553 58333	−.01377 25000	.01237 48625	.70
.71	−.00075 92562	−.00551 65929	−.01380 63071	.01192 34582	.71
.72	−.00020 16000	−.00548 20864	−.01380 07296	.01147 15066	.72
.73	+.00035 72438	−.00543 25742	−.01375 59457	.01101 94352	.73
.74	.00091 39000	−.00536 83448	−.01367 22005	.01056 76581	.74
.75	.00146 48438	−.00528 97135	−.01354 98047	.01011 65771	.75
.76	.00200 64000	−.00519 70219	−.01338 91328	.00966 65813	.76
.77	.00253 47438	−.00509 06362	−.01319 06220	.00921 80472	.77
.78	.00304 59000	−.00497 09469	−.01295 47704	.00877 13394	.78
.79	.00353 57438	−.00483 83675	−.01268 21356	.00832 68101	.79
.80	.00400 00000	−.00469 33333	−.01237 33333	.00788 48000	.80
.81	.00443 42438	−.00453 63008	−.01202 90356	.00744 56378	.81
.82	.00483 39000	−.00436 77464	−.01164 99696	.00700 96408	.82
.83	.00519 42438	−.00418 81655	−.01123 69159	.00657 71149	.83
.84	.00551 04000	−.00399 80715	−.01079 07072	.00614 83549	.84
.85	.00577 73438	−.00379 79948	−.01031 22266	.00572 36443	.85
.86	.00598 99000	−.00358 84819	−.00980 24061	.00530 32562	.86
.87	.00614 27438	−.00337 00941	−.00926 22255	.00488 74524	.87
.88	.00623 04000	−.00314 34069	−.00869 27104	.00447 64848	.88
.89	.00624 72438	−.00290 90088	−.00809 49308	.00407 05945	.89
.90	.00618 75000	−.00266 75000	−.00747 00000	.00367 00125	.90
.91	.00604 52438	−.00241 94921	−.00681 90725	.00327 49598	.91
.92	.00581 44000	−.00216 56064	−.00614 33429	.00288 56476	.92
.93	.00548 87438	−.00190 64734	−.00544 40445	.00250 22771	.93
.94	.00506 19000	−.00164 27315	−.00472 24472	.00212 50402	.94
.95	.00452 73438	−.00137 50260	−.00397 98568	.00175 41193	.95
.96	.00387 84000	−.00110 40085	−.00321 76128	.00138 96876	.96
.97	.00310 82438	−.00083 03354	−.00243 70874	.00103 19091	.97
.98	.00220 99000	−.00055 46669	−.00163 96837	.00068 09388	.98
.99	.00117 62438	−.00027 76667	−.00082 68344	.00033 69233	.99
1.00	.00000 00000	.00000 00000	.00000 00000	.00000 00000	1.00

TABLE IV: COEFFICIENTS OF SEVENTH ORDER TERMS

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta} + \text{4th order terms} + \text{5th order terms} + \text{6th order terms} + I(m)\left(\frac{\Delta^3}{\Delta}\right)^2\frac{\Delta^3}{\Delta} + J(m)\frac{\Delta^3}{\Delta}\frac{\Delta^4}{\Delta} + K(m)\frac{\Delta^3}{\Delta}\frac{\Delta^5}{\Delta} + L(m)\frac{\Delta^7}{\Delta} + \text{8th order terms} + \cdots$$

m	$I(m)$	$J(m)$	$K(m)$	$L(m)$	m
0	.00000 00000	.00000 00000	.00000 00000	.00000 00000	0
.1	−.01094 62500	−.00964 60625	−.01207 56375	−.01109 36963	.1
.2	−.00528 00000	−.00972 80000	−.01291 94667	−.01692 67200	.2
.3	.00443 62500	−.00540 70625	−.00805 32375	−.01895 72738	.3
.4	.01104 00000	−.00008 53333	−.00136 32000	−.01837 05609	.4
.5	.01171 87500	.00423 17708	.00462 23958	−.01611 32813	.5
.6	.00696 00000	.00660 80000	.00847 68000	−.01292 54400	.6
.7	−.00049 87500	.00688 37708	.00963 92625	−.00936 95388	.7
.8	−.00672 00000	.00541 86667	.00820 05333	−.00585 72800	.8
.9	−.00761 62500	.00286 89375	.00470 93625	−.00267 38663	.9
1.0	.00000 00000	.00000 00000	.00000 00000	.00000 00000	1.0

TABLE V: COEFFICIENTS OF EIGHTH ORDER TERMS

$$x = x_0 + ph;\ [f(x) - f(x_0)]/\Delta = m;\ p = m + \frac{m(1-m)}{2}\frac{\Delta^2}{\Delta} + \frac{m(1-m)(m-2)}{6}\frac{\Delta^3}{\Delta} + \text{4th order terms} + \text{5th order terms} + \text{6th order terms} + \text{7th order terms} + M(m)\left(\frac{\Delta^3}{\Delta}\right)^4 + N(m)\frac{\Delta^3}{\Delta}\left(\frac{\Delta^3}{\Delta}\right)^3 + O(m)\left(\frac{\Delta^3}{\Delta}\right)^2\frac{\Delta^4}{\Delta} + P(m)\left(\frac{\Delta^4}{\Delta}\right)^2 + Q(m)\frac{\Delta^3}{\Delta}\frac{\Delta^5}{\Delta} + R(m)\frac{\Delta^2}{\Delta}\frac{\Delta^6}{\Delta} + S(m)\frac{\Delta^8}{\Delta} + \cdots$$

m	$M(m)$	$N(m)$	$O(m)$	$P(m)$	m
0	.00000 00000	.00000 00000	.00000 00000	.00000 00000	0
.1	.00166 50000	.00642 97500	.00755 88750	.00342 30875	.1
.2	−.00072 00000	.00207 28889	.00287 46667	.00320 32000	.2
.3	−.00246 75000	−.00363 47500	−.00383 51250	.00152 61750	.3
.4	−.00204 00000	−.00646 40000	−.00760 80000	−.00030 50667	.4
.5	.00000 00000	−.00564 23611	−.00716 14583	−.00162 76042	.5
.6	.00204 00000	−.00238 40000	−.00352 80000	−.00221 76000	.6
.7	.00246 75000	.00130 02500	.00109 98750	−.00212 78833	.7
.8	.00072 00000	.00351 28889	.00431 46667	−.00156 05333	.8
.9	−.00166 50000	.00309 97500	.00422 88750	−.00077 09625	.9
1.0	.00000 00000	.00000 00000	.00000 00000	.00000 00000	1.0

m	$Q(m)$	$R(m)$	$S(m)$	m
0	.00000 00000	.00000 00000	.00000 00000	0
.1	.00740 61525	.00974 08950	.00956 83130	.1
.2	.00707 07200	.00999 50933	.01438 77120	.2
.3	.00352 32925	.00578 82650	.01587 67168	.3
.4	−.00047 10400	.00044 60800	.01515 57120	.4
.5	−.00345 05208	−.00403 64583	.01309 20410	.5
.6	−.00486 97600	−.00667 32800	.01034 03520	.6
.7	−.00478 13675	−.00721 03150	.00737 85118	.7
.8	−.00358 05867	−.00589 05600	.00453 93920	.8
.9	−.00180 73275	−.00325 83450	.00203 88230	.9
1.0	.00000 00000	.00000 00000	.00000 00000	1.0

THE TENSOR REPRESENTATION OF THE FIGURES OF STUDY'S "GEOMETRIE DER DYNAMEN"

By Domina Eberle Spencer

In his important work, "Geometrie der Dynamen," Eduard Study has considered a number of important geometric figures which have noteworthy applications in mechanics. The simplest of these figures is the vector. Vector analysis has received universal acclaim in physics and engineering and it seems probable that some of the more complicated figures of Study may likewise have important applications. The investigation of the subject has been handicapped in the past in several ways. The greatest difficulty has been the lack of systematic classification under the affine and metric groups. Since Study did not clearly define the group under which his figures have meaning, he was not led to a convenient notation. By a systematic consideration of the groups under which these figures are defined and the tensor notation belonging to these groups, the previous difficulties tend to disappear and the whole subject takes on new life and interest.

An earlier paper,[1] "Geometric Figures in Affine Space", has treated the affine ancestors of the Study figures and has thus laid the foundation for the tensor representation of the figures themselves. The present paper continues the work by the introduction of the metric and the detailed study of the various figures.

The investigation discloses considerable overlapping among the figures of "Geometrie der Dynamen." Such overlapping is eliminated in the revised set proposed in this paper. It is believed that the systematic approach to the subject given in these two papers will make possible important applications which were obscured by the previous unsatisfactory development.

Introduction of the Metric

The introduction of a metric into affine space permits the geometric figures to have associated with them, not only a support and a polarization, but also a scalar magnitude (the metric normalization). It also gives a number of new correspondences between the descendants of the figures of affine space. If the support, polarization, and metric normalization are known, then the tensor coordinates of the figure are completely defined.

The metric space is obtained by introduction into the affine space of a metric tensor $g^{\kappa\lambda}$. This tensor permits comparison of lengths in different directions and of angles measured about different axes. Affine normalizations can be replaced by lengths and angles.

There is an infinite number of ways in which the tensor $g^{\kappa\lambda}$ can be defined. Of this infinitude we take the simplest one, the *Euclidean Metric.* Single out a symmetrical tensor $g^{\kappa\lambda}$ of rank three, satisfying the condition,

$$g^{\kappa 0} a_0 = 0.$$

[1] Presented at the Poughkeepsie meeting of the American Mathematical Society, September, 1942. Journal of Mathematics and Physics, 23, 1944, p. 1.

Introduce the further specialization of identical scales on orthogonal coordinate axes and a rotation geometry ($\Delta = +1$). Then the coordinate matrix of $g^{\alpha\lambda}$ may be defined by

$$g^{\alpha\lambda} = \begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}. \tag{1}$$

Since this matrix is of rank three, it does not have a unique inverse.

It will be convenient to consider also the tensor g^{ik} whose matrix is a submatrix of Eq. (1). This tensor has a unique inverse, and

$$g_k^i \overset{*}{=} g^{ik} \overset{*}{=} \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} \overset{*}{=} g_{ik}, \qquad g^{ik}g_{kl} = \delta_l^i. \tag{2}$$

The metric tensor permits raising and lowering of indices and allows correspondences between contravariant and covariant figures, relations that are not found in affine space. Combination of the metric tensor with the figures of affine space, which were discussed in the previous paper[1], gives the tensor representation of all of the figures of Study.

The Stab

The *Stab* is defined by Study as the figure of two proper points. Its support is the line joining the two points and its polarization is the order in which the two points are taken, while its magnitude is the distance between the points. Two Stäbe are defined as equal if they lie on the same support and if their endpoints can be brought into coincidence by translation of one Stab along the common line of support.

The affine ancestor of the Stab of Study is the affine figure of two proper points[2] defined by

$$p^{\alpha\lambda} = x^{[\alpha}y^{\lambda]}. \tag{3}$$

This is a figure of two proper points having the same support and polarization as the Stab. A scalar magnitude may be associated with it as follows:

$$\bar{p}^{\alpha\lambda} = \sqrt{g_{ij}p^{0i}p^{0j}}. \tag{4}$$

However, this scalar has in general no simple geometric meaning. To obtain the tensor representation of the Stab, the foregoing tensor must be multiplied by a scalar factor, giving

$$S^{\alpha\lambda} = \frac{x^{[\alpha}y^{\lambda]}}{x^0y^0}. \tag{5}$$

The new figure has the same support and polarization as $p^{\alpha\lambda}$, while its normalization is the distance between the points x^α and y^α,

$$\bar{S}^{\alpha\lambda} = \sqrt{g_{ij}S^{0i}S^{0j}} = \frac{1}{x^0y^0}\sqrt{(x^0y^1 - x^1y^0)^2 + (x^0y^2 - x^2y^0)^2 + (x^0y^3 - x^3y^0)^2}. \tag{6}$$

[2] The *proper rhabdel* of Reference 1.

Since, by the definition of the Stab, both the points are proper, the metric normalization must always remain finite. The null Stab has the metric normalization zero.

The Stäbepaare

The *Stäbepaare* may be defined as the sum of two parallel complanar Stäbe. An axial vector is associated with every Stäbepaare, and vice versa. If a definite sense of rotation is introduced into the plane containing the Stäbepaare, the parallelogram formed by the component Stäbe has a fully defined positive or negative area. This area (and the Stäbepaare) remains unchanged when the lengths of the two Stäbe are changed in reciprocal ratio, or when the figure is rotated in its plane or moved parallel to itself. The area associated with the Stäbepaare is called its *moment*. The associated vector is defined so that its direction is normal to the plane of the Stäbepaare. The polarization is associated in a 1:1 way with that of the plane, and the length of the vector is equal to the moment.

The affine ancestor of the Stäbepaare is the affine figure of two free vectors.[3] In metric space, this geometric figure whose tensor representation is

$$S^{ij} = v^{[i}w^{j]}, \tag{7}$$

becomes identical with the Stäbepaare. The support and polarization of the two are identical. Their metric normalization is

$$\begin{aligned} S^{ij} &= \sqrt{g^{ij}E_{0ikl}S^{kl}E_{0jab}S^{ab}} \\ &= \sqrt{(v^1w^2 - v^2w^1)^2 + (v^1w^3 - v^3w^1)^2 + (v^2w^3 - v^3w^2)^2}. \end{aligned} \tag{8}$$

Evidently Eq. (8) gives the area of the parallelogram determined by the free vectors v^i and w^i, which is the moment of the Stäbepaare. Since the free vectors are of finite length, the metric normalization must always remain finite. If the normalization becomes zero, the Stäbepaare reduces to the null Stab.

A 1:1 correspondence can be set up between a Stäbepaare S^{ij} and a free vector v^i by the following relations:

$$\begin{aligned} &S^{ij}E_{0ijg}g^{lk} = v^k \\ &S^{ij} = E^{0kij}g_{kl}v^l, \qquad v^1 = S^{23}, \qquad v^2 = S^{31}, \qquad v^3 = S^{12}. \end{aligned} \tag{9}$$

The supports of free vector and Stäbepaare are mutually orthogonal, and together they form an improper *Linienkreuz* of Study. When the screw sense of the space is given, polarizations correspond. Their metric normalizations are then equal, by definition.

The Proper Keil

The *proper Keil* is defined by Study as the figure formed by two planes that are not perpendicular, nor parallel unless coincident. The line of intersection

[3] The *improper rhabdel* of 1.

is the support of this figure, and its polarization is the order in which the planes are taken. Study defines the magnitude (called the *opening*) of the Keil as the tangent of the angle between the planes. Two Keile are defined as equal if they have the same support and polarization and if their planes can be brought into coincidence by rotation about the support.

The affine ancestor of the proper Keil is the affine figure of two proper planes.[4] It is defined by the equation,

$$p_{\alpha\lambda} = u_{[\alpha}v_{\lambda]}. \tag{10}$$

This is a figure of two proper planes having the same support and polarization as the proper Keil. The metric normalization associated with the proper strophel is defined by

$$\bar{p}_{\alpha\lambda} = \sqrt{g_{ij}E^{0ikl}p_{kl}E^{0jab}p_{ab}} \tag{11}$$

a scalar which has no specific geometric meaning. To obtain the tensor representation of the proper Keil, one must find a similar figure whose normalization is the tangent of the angle between the two planes. Such a figure is

$$K_{\alpha\lambda} = \frac{u_{[\alpha}v_{\lambda]}}{g^{ij}u_iv_j}. \tag{12}$$

Its normalization is the tangent of the angles between the planes, or

$$\begin{aligned}\bar{K}_{\alpha\lambda} &= \sqrt{g_{ij}E^{0ikl}K_{kl}E^{0jab}K_{ab}} \\ &= \frac{\sqrt{(u_1v_2 - u_2v_1)^2 + (u_1v_3 - u_3v_1)^2 + (u_2v_3 - u_3v_2)_2}}{u_1v_1 + u_2v_2 + u_3v_3}.\end{aligned} \tag{13}$$

The proper Keil differs also from the affine figure of two proper planes[4] in that its planes may not be perpendicular without making the normalization infinite. The null Keil has the metric normalization zero.

The Improper Keil

The *improper Keil* is the geometric figure defined by Study in terms of two parallel planes. Its support is the direction normal to the planes and its polarization is the order of the planes. Its magnitude (the *Parting*) is the separation of the planes. A vector can be associated with it in a 1:1 way. The direction of the vector is normal to the planes, its polarization is uniquely associated with that of the improper Keil, and its length is equal to the parting of the Keil.

The affine ancestor of the improper Keil is the affine figure of a free and an improper plane or of two non-coincident parallel proper planes.[5] If the coordinates are defined in the latter way,

$$p_{\alpha\lambda} = u_{[\alpha}v_{\lambda]} \quad \text{if} \quad u_\alpha = \lambda v_k\,. \tag{14}$$

Thus,

$$p_{0i} \not\equiv 0, \qquad p_{kl} \equiv 0.$$

[4] The *proper strophel* of 1.

[5] The *semi-free strophel* of 1.

The support is a free plane direction and the polarization is the same as that of the improper Keil. The magnitude of the figure is

$$\bar{p}_{0i} = \sqrt{g^{ij} p_{0i} p_{0j}} \,. \tag{15}$$

The tensor representation of the improper Keil is obtained by multiplying this figure by a scalar factor, or

$$K_{\kappa\lambda} = \frac{u_{[\kappa} v_{\lambda]}}{g^{ij} u_i v_j} \tag{16}$$

where $u_k = \lambda v_k$ and $p_{kl} \equiv 0$.
Thus the metric normalization is

$$\begin{aligned} \bar{K}_{0i} &= \sqrt{g^{ij} p_{0i} p_{0j}} \\ &= \frac{\sqrt{(u_0 v_1 - u_1 v_0)^2 + (u_0 v_2 - u_2 v_0)^2 + (u_0 v_3 - u_3 v_0)^2}}{u_1 v_1 + u_2 v_2 + u_3 v_3} \end{aligned} \tag{17}$$

which is the parting or separation of the parallel planes. The null Keil is obtained when the two planes are coincident.

The Motor, Rotor, and Translator

The *Motor* of Study is the geometric figure representing the general sum of Stäbe or Keile. The Motor can be defined in terms of two lines, with the restriction that both be proper and they do not cross one another at right angles. The support is the common normal of the two lines, and the polarization is the order of the lines. The Motor has both a contravariant and a covariant representation.

Let us first consider the contravariant representation. The affine ancestor of the contravariant representation is the proper affine figure of sums of figures of points.[6] If the length of the Motor is defined as the separation of the two lines, the restriction that both lines be proper corresponds to the restriction that the Stäbepaare of the decomposition be of finite metric normalization. If the opening is defined as the tangent of the angle between the lines, the restriction that they do not cross at right angles corresponds to the restriction that the length of the proper Stab be finite.

By the methods described in the previous paper[1], the decomposition of the Motor $M^{\kappa\lambda}$ can be effected. $S^{\kappa\lambda}$ has the components

$$S^{\kappa\lambda} \begin{cases} S^{0i} = M^{0i} \\ S^{kl} = M^{kl} - \lambda E^{0ikl} g_{ij} M^{0j} \end{cases} \tag{18}$$

and the Stäbepaare component S^{kl} is

$$S^{kl} = \lambda E^{0ikl} g_{ij} M^{0j} \tag{19}$$

where

$$\lambda = \frac{M^{01}M^{23} + M^{02}M^{13} + M^{03}M^{12}}{(M^{01})^2 + (M^{02})^2 + (M^{03})^2} \,.$$

[6] The *proper kineal* of 1.

The support of the proper Stab is the same as the support of the Motor. In terms of two lines (X, Y),

$$\text{Length of Motor} = \text{dist}\,(X, Y)\ \bar{S}^{kl} = \frac{M^{01}M^{23} + M^{02}M^{13} + M^{03}M^{12}}{\sqrt{(M^{01})^2 + (M^{02})^2 + (M^{03})^2}}. \tag{20}$$

$$\text{Opening of Motor} = \tan \text{ang}\,(X, Y) = \bar{S}^{\omega\lambda} = \sqrt{(M^{01})^2 + (M^{02})^2 + (M^{03})^2}. \tag{21}$$

The restrictions can be written in symbols as follows:

(1) X and Y proper corresponds to $\bar{S}^{kl}$ finite.
(2) X and Y not orthogonal corresponds to $\bar{S}^{\omega\lambda}$ finite. (22)

The special cases are

(1) $$\bar{S}^{\omega\lambda} = 0$$
$$\bar{S}^{kl} = \sqrt{(M^{12})^2 + (M^{13})^2 + (M^{23})^2} \tag{23}$$

which is the *Translator* of Study

(2) $$\bar{S}^{kl} = 0$$
$$\bar{S}^{\omega\lambda} = \sqrt{(M^{01})^2 + (M^{02})^2 + (M^{03})^2} \tag{24}$$

which is the *Rotor* of Study.

Now take the covariant representation of the Motor, the affine ancestor in this case being the proper affine figure of sums of figures of planes.[7] The restriction that both lines be proper corresponds to the restriction that the planes defining the proper Keil be both proper or that the metric normalization of the Keil be finite. The restriction that the lines do not intersect at right angles corresponds to the restrictions that the planes of the proper Keil do not intersect at right angles.

The decomposition of the covariant representation of the Motor $M_{\omega\lambda}$ has the proper Keil component $K_{\omega\lambda}$.

$$K_{\omega\lambda}\begin{cases} K_{0i} = M_{0i} - \lambda g_{ij}E^{0jkl}M_{kl} \\ K_{ij} = M_{ij}. \end{cases} \tag{25}$$

The improper Keil component K_{0i} is

$$K_{0i} = \lambda g_{ij}E^{0jkl}M_{kl} \tag{26}$$

where

$$\lambda = \frac{M_{01}M_{23} + M_{02}M_{13} + M_{03}M_{12}}{(M_{23})^2 + (M_{13})^2 + (M_{12})^2}.$$

[7] The *proper helissel* of 1.

The support of the proper Keil is identical with the support of the Motor. In terms of two lines (X, Y),

$$\text{Length of Motor} = \text{dist}\,(X, Y) = \bar{K}_{0i}$$

$$= \frac{M_{01}M_{23} + M_{02}M_{13} + M_{03}M_{12}}{(M_{12})^2 + (M_{13})^2 + (M_{23})^2}. \quad (27)$$

$$\text{Opening of Motor} = \tan \text{ang}\,(X, Y) = \bar{K}_{\infty\infty}$$

$$= \sqrt{(M_{12})^2 + (M_{13})^2 + (M_{23})^2}. \quad (28)$$

In symbols, the restrictions can be written

(1) X and Y proper corresponds to $\bar{K}_{0i}$ finite.
(2) X and Y not orthogonal, corresponds to $\bar{K}_{\infty\infty}$ finite. (29)

The special cases are

(1) $$\bar{K}_{\infty\infty} = 0$$

$$\bar{K}_{0i} = \sqrt{(M_{01})^2 + (M_{02})^2 + (M_{03})^2} \quad (30)$$

which is the *Translator* of Study.

(2) $$\bar{K}_{0i} = 0$$

$$\bar{K}_{\infty\infty} = \sqrt{(M_{12})^2 + (M_{13})^2 + (M_{23})^2} \quad (31)$$

which is the *Rotor* of Study.

If the two lines of a Motor intersect at a proper point, the length of the Motor is zero and the figure is called a *Rotor*. Note that its decomposition then degenerates into a proper Stab or proper Keil.

If the lines of the Motor are parallel, the opening of the Motor vanishes and the figure is called a *Translator*. In this case, the decomposition degenerates into a Stäbepaare or an improper Keil.

The Proper Quirl

Study takes as the *proper Quirl* the figure of a proper point and a proper plane that do not coincide. These elements are defined as the initial point and the final plane or as the initial plane and the final point of the Quirl. The support is the line perpendicular to the plane and passing through the point; and the Quirl is unchanged if point and plane move along the line, their polarization and separation remaining unchanged. The *length* of the Quirl is the reciprocal of the distance between the point and the plane. A Stab and a proper Keil having the same support, a corresponding polarization, and the same normalization are uniquely associated with the proper Quirl. If the proper point goes to infinity, the null Quirl is obtained.

The affine ancestor of the proper Quirl is the affine figure of a proper point and plane[8]

$$p^{\kappa}_{\cdot\lambda} = x^{\kappa} u_{\lambda} . \tag{32}$$

The metric normalization associated with it is

$$\bar{p}^{\kappa}_{\cdot\lambda} = \left| \frac{p^{\kappa}_{\cdot\lambda}}{\sqrt{g^{ij}\, p^{0}_{\cdot i} p^{0}_{\cdot j}}} \right| \tag{33}$$

which is not a quantity with simple geometric meaning.

The tensor representation of the proper Quirl is obtained by expressing a proper figure of points[2] and a proper figure of planes[4] in terms of the proper figure of a point and a plane.[5] A free vector is associated with this figure. This vector is normal to the proper plane, has a corresponding polarization, and its length is equal to the reciprocal of the normalization of the proper figure of a point and a plane.[6]

The vector is defined by the equation,

$$\begin{aligned} v^{i} &= \left| \frac{1}{p^{\kappa}_{\cdot\kappa}} \right| g^{ij} p^{0}_{\cdot j} \\ &= \left| \frac{1}{p^{0}_{\cdot 0} + p^{1}_{\cdot 1} + p^{2}_{\cdot 2} + p^{3}_{\cdot 3}} \right| p^{0}_{\cdot i} . \end{aligned} \tag{34}$$

Similarly, a free plane is associated with the figure of a point and a plane[5] by

$$w^{i} = \left| \frac{1}{p^{\kappa}_{\cdot\kappa}} \right| p^{0}_{\cdot i} . \tag{35}$$

A proper figure of points may be associated with the proper figure of a point and a plane[5] by defining this figure so that it will have the direction of the vector v^{i} and will pass through the proper point x^{κ}. This gives one tensor representation of the proper Quirl:

$$Q^{\kappa\lambda} = \frac{p^{[\kappa}_{\cdot 0} B^{\lambda]}_{1}\, g^{ij}\, p^{0}_{\cdot j}}{|\, p^{\kappa}_{\cdot\kappa}\,|\, p^{0}_{\cdot 0}} . \tag{36}$$

A proper figure of planes[4] may also be associated with the proper figure of a point and a plane[5] by choosing the proper figure of planes[4] associated with the figure of Eq. (36), by the method of the previous paper:

$$Q_{\kappa\lambda} = \frac{E_{\kappa\lambda\mu\nu} p^{[\mu}_{\cdot 0} B^{\nu]}_{1}\, g^{ij}\, p^{0}_{\cdot j}}{|\, p^{\kappa}_{\cdot\kappa}\,|\, p^{0}_{\cdot 0}} . \tag{37}$$

This gives a covariant tensor representation of the proper Quirl

Equations (36) and (37) are the analytic representation of the proper Quirl of Study. The figure is defined in terms of a proper point and a proper plane. The support is normal to the proper plane and contains the proper point. The

[8] The *proper charybdel* of 1.

magnitude is the reciprocal of the separation of the proper point and the proper plane, and the same figure is defined by all suitably ordered proper points on the support and planes normal to the support, all of equal separation. Thus, many proper figures of points and planes[8] which have the same polarization and normalization and which are associated with the same line support define the same proper Quirl.

The Improper Quirl

The *improper Quirl* of Study is the figure of a proper plane and an improper point not in the plane. If the point is in the plane, the figure consists of a proper plane and a group of parallel lines making an angle different from zero with the plane. The support of the improper Quirl is the set of parallel lines orthogonal to the proper plane. Its polarization is the order of proper plane and improper point. The normalization is the cotangent of the angle between proper plane and the set of parallel lines representing the improper point. If the improper point lies on the support, the null Quirl is obtained.

The affine ancestor of the improper Quirl is the figure of a proper plane and an improper point[9] whose tensor representation is

$$p^i_{.\lambda} = x^i u_\lambda . \tag{38}$$

The normalization is the tangent of the angle between the direction and the proper plane, or

$$\tilde{p}^i_{.\lambda} = \frac{|p^i_{.i}|}{\sqrt{g_{ij} p^i_{.0} p^j_{.0} - (p^i_{.i})^2}} . \tag{39}$$

The tensor interpretation of Study's figure is found by defining a free vector and a free plane in terms of the figure of a proper plane and an improper point.[9] The free vector is normal to the plane of the semi-improper figure[9] and its length is the reciprocal of the normalization of the same figure. Therefore, the contravariant tensor representation of the improper Quirl is

$$Q^i = \left| \frac{\sqrt{g_{ij} p^i_{.0} p^j_{.0} - (p^i_{.i})^2}}{p^i_{.i}} \right| \frac{g^{ij} p^1_{.j}}{\sqrt{g^{ij} p^1_{.i} p^1_{.j}}} . \tag{40}$$

Similarly, the free plane associated with this figure is

$$Q_i = \left| \frac{\sqrt{g_{ij} p^i_{.0} p^j_{.0} - (p^i_{.i})^2}}{p^i_{.i}} \right| \frac{p^1_{.i}}{\sqrt{g^{ij} p^1_{.i} p^1_{.j}}} . \tag{41}$$

This is the covariant tensor representation of the improper Quirl.

The figure represented analytically by Eqs. (40) and (41) is the improper Quirl of Study. It is defined in terms of a proper and an improper point. The magnitude is the reciprocal of the normalization of the associated figure of a proper plane and an improper point.[9] The same figure is defined by all suitably

[9] The *semi-improper charybdel* of 1.

ordered free vectors and proper planes of equal angle. Thus there are many affine ancestors defining the same improper Quirl. All such figures which have the same polarization and normalization and which are associated with the same free support define the same improper Quirl.

The Impulsor, Torsor, and Ejector

The *Impulsor*, is defined as the generalized sum of Quirle. It can therefore be represented by a contravariant or by a covariant alternating tensor of valence two, which does not in general satisfy Plücker's relation. A decomposition can always be made into a proper Quirl and an improper Quirl.

It is now possible to show how the Impulsor can be defined in terms of two lines, as was done by Study. The restrictions on the two lines are that at least one of them must be proper and that they must not lie in the same plane. The length of the Impulsor is defined as the separation of the two lines. The restriction that the lines must not lie in the same plane corresponds to the restriction that the point and plane of the proper Quirl must not coincide. If the opening is defined as the tangent of the angle between the lines, the restriction that at least one of them be proper corresponds to the restriction that the line and plane of the improper Quirl must not coincide.

If the two lines of the Impulsor are parallel, the length of the Impulsor is infinite and the Impulsor is called a *Torsor*. The decomposition then degenerates into a proper Quirl.

If one of the lines of the Impulsor is proper and one improper, the opening of the Impulsor is infinite and the figure is called an *Ejector*. Then the decomposition degenerates into an improper Quirl.

As with the Motor, a decomposition can be written for the Impulsor $I^{\alpha\lambda}$ or $I_{\alpha\lambda}$. The proper Quirl component ($Q^{\alpha\lambda}$ or $Q_{\alpha\lambda}$) has the coordinates,

$$Q^{\alpha\lambda}\begin{cases} Q^{0i} = I^{0i} \\ Q^{kl} = I^{kl} - \underset{(1)}{\lambda} E^{0ikl} g_{ij} I^{0j} \end{cases} \qquad Q_{\alpha\lambda}\begin{cases} Q_{0i} = I_{0i} - \underset{(2)}{\lambda} E^{0jkl} I_{kl} g_{ij} \\ Q_{ij} = I_{ij} \end{cases} \tag{42}$$

The improper Quirl (Q^i or Q_i) has the coordinates,

$$\begin{aligned} Q^i &= \underset{(1)}{\lambda} I^{0i} \\ Q_i &= \underset{(2)}{\lambda} g_{ij} E^{0jkl} I_{kl} \end{aligned} \tag{43}$$

where

$$\begin{aligned} \underset{(1)}{\lambda} &= \frac{I^{01} I^{23} + I^{02} I^{13} + I^{03} I^{12}}{(I^{01})^2 + (I^{02})^2 + (I^{03})^2} \\ \underset{(2)}{\lambda} &= \frac{I_{01} I_{23} + I_{02} I_{13} + I_{03} I_{12}}{(I_{23})^2 + (I_{13})^2 + (I_{12})^2} . \end{aligned} \tag{44}$$

The support of the proper Quirl is that of the Impulsor.

In terms of two lines (X, Y),

$$\begin{aligned}\text{Length of Impulsor} = \text{dist}\,(X, Y) &= 1/\bar{Q}^{\omega\lambda} = 1/\sqrt{(I^{01})^2 + (I^{02})^2 + (I^{03})^2} \\ &= 1/\bar{Q}_{\omega\lambda} = 1/\sqrt{(I_{12})^2 + (I_{13})^2 + (I_{23})^2}\end{aligned} \tag{45}$$

$$\begin{aligned}\text{Opening of Impulsor} = \tan \text{ang}\,(X, Y) &= 1/\bar{Q}^{i} \\ &= \frac{\sqrt{(I^{01})^2 + (I^{02})^2 + (I^{03})^2}}{I^{01}I^{23} + I^{02}I^{13} + I^{03}I^{12}} \\ &= 1/\bar{Q}_i = \frac{\sqrt{(I_{12})^2 + (I_{13})^2 + (I_{23})^2}}{I_{01}I_{23} + I_{02}I_{13} + I_{03}I_{12}}\end{aligned} \tag{46}$$

The restriction can be written symbolically as follows:

(1) X and Y are not both improper corresponds to $\bar{Q}^{\omega\lambda}$ and $\bar{Q}_{\omega\lambda}$ finite.
(2) X and Y do not lie in the same plane corresponds to $\bar{Q}^{i}$ and $\bar{Q}_i$ finite. (47)

The special cases are

(1) $\bar{Q}^{\omega\lambda} = \bar{Q}_{\omega\lambda} = 0$

$$\begin{aligned}\bar{Q}^{i} &= \sqrt{(I^{12})^2 + (I^{13})^2 + (I^{23})^2} \\ &= \bar{Q}_i = \sqrt{(I_{01})^2 + (I_{02})^2 + (I_{03})^2},\end{aligned} \tag{48}$$

which is the *Ejector* of Study.

(2) $\bar{Q}^{i} = \bar{Q}_i = 0.$

$$\begin{aligned}\bar{Q}^{\omega\lambda} &= \sqrt{(I^{01})^2 + (I^{02})^2 + (I^{03})^2} \\ \bar{Q}_{\omega\lambda} &= \sqrt{(I_{12})^2 + (I_{13})^2 + (I_{23})^2},\end{aligned} \tag{49}$$

which is the *Torsor* of Study.

This completes the tensor representation of the figures of Study.

A System of Figures Proposed for Future Investigations

The following system of geometric figures contains the significant free and attached vectorial figures of metric space. It would be possible to use the figures of Study without modification, but this would introduce non-essential difficulties into the definitions. The figures of Study are defined in terms of points, lines, and planes; but *the essentials are the support, the polarizations, and the normalization.*

On this basis, it is more convenient to define six new terms:

(1) Proper rhabdor
(2) Improper Rhabdor
(3) Kineor
(4) Proper charybdor
(5) Improper charybdor
(6) Dynator.

The Proper Rhabdor

The proper rhabdor is represented by an alternating contravariant tensor $p^{\omega\lambda}$ of valence two satisfying Plücker's relation,

$$p^{[\omega\lambda}p^{\mu\nu]} = 0.$$

Support = line $\bar{p}^{\alpha\lambda}$.
Polarization = sense along or about support.
Magnitude = $\sqrt{(p^{01})^2 + (p^{02})^2 + (p^{03})^2}$
Restriction, magnitude must be finite.

The Improper Rhabdor

The improper rhabdor is similar to the proper rhabdor, but $p^{0i} = 0$. The tensor representation is p^{ij}.

Support = free plane $\bar{p}^{ij}$.
Polarization = sense of rotation in plane or sense along perpendicular to plane.
Magnitude = $\sqrt{(p^{12})^2 + (p^{13})^2 + (p^{23})^2}$
Restriction, magnitude must be finite.

The Kineor

The general sum of rhabdors is a kineor. The kineor is an alternating contravariant bivalent tensor $p^{\alpha\lambda}$ of valence two, which does not in general satisfy Plücker's relation. It can be decomposed into proper and improper rhabdors with respect to any proper points as follows:

$$\begin{aligned} q^{\alpha\lambda} &= x^{[\alpha} p^{|0|\lambda]}/x^0 \\ r^{kl} &\to p^{\alpha\lambda} - x^{[\alpha} p^{|0|\lambda]}/x^0 \end{aligned} \tag{50}$$

The Proper Charybdor

The proper charybdor is represented by an alternating contravariant tensor $p^{\alpha\lambda}$ of valence two, which satisfies Plücker's equation,

$$p^{[\alpha\lambda} p^{\mu\nu]} = 0.$$

Support = line $\bar{p}^{\alpha\lambda}$.
Polarization = sense along or about support.
Magnitude = $1/\sqrt{(p^{01})^2 + (p^{02})^2 + (p^{03})^2}$
Restriction, magnitude must be finite.

The Improper Charybdor

The improper charybdor is like the proper charybdor except that $p^{0i} = 0$. Its tensor representation is p^{ij}.

Support = free plane $\bar{p}^{ij}$.
Polarization = sense of rotation in plane or sense along perpendicular to plane.
Magnitude = $1/\sqrt{(p^{12})^2 + (p^{13})^2 + (p^{23})^2}$
Restriction, magnitude must be finite.

The Dynator

The dynator is the general sum of charybdors and is represented by a contravariant alternating bivalent tensor $p^{\alpha\lambda}$, which does not in general satisfy Plücker's

relation. It can be decomposed with respect to any proper point into a proper and an improper charybdor:

$$\begin{aligned} q^{\omega\lambda} &= x^{[\kappa}p^{|0|\lambda]}/x^0 \\ r^{kl} &\to p^{\omega\lambda} - x^{[\kappa}p^{|0|\lambda]}/x^0. \end{aligned} \tag{51}$$

The foregoing set of six figures will be useful in practical applications to mechanics and other branches of physics. With these figures it will be possible to develop a discipline analogous to vector analysis. Such a development would be very difficult on the basis of the original Study figures, but with the new set it is perfectly feasible.

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A BOUNDARY VALUE PROBLEM IN PLANE PLASTICITY FOR THE COULOMB YIELD CONDITION*

By N. Coburn

1. **Introduction.** We shall consider a semi-infinite plane $x > 0$, composed of plastic material which follows the Coulomb yield condition. The normal stresses σ_x, σ_y and the shear stress σ_{xy} acting along $x = 0$ will be considered as known. Our problem will be to determine the stresses acting at any point in the interior of the semiplane $x > 0$.

The Coulomb yield condition in the plane plasticity problem has been investigated both geometrically and analytically. The geometry of the slip lines for this yield condition has been studied by W. Hartmann, A. Nadai [1],[1] and I. Capuano [2]. Further, the geometry of the lines of maximum shearing stress for the same yield condition has been investigated by the author [3]. The analytical determination of the stresses depends upon the solution of a second order nonlinear partial differential equation.[2] Several particular solutions of this differential equation have been obtained [4]. Furthermore, W. Sokolovsky [5] has shown that a line of rupture exists for a plane material obeying this yield condition.

Our approach to the problem will be analytic. In particular, we shall modify a method used in studying the stress distribution in a semi-infinite plane $x > 0$ for a perfectly plastic material [6]. First, we expand the stresses σ_x, σ_y, σ_{xy}, acting at any point and the functions $\sin 2\gamma$, $\cos 2\gamma$ (where γ is the angle between the x-axis and the tangent to a line of maximum shearing stress at the point under consideration) in power series of the friction coefficient. Substituting these power series into Levy's equations,[3] we obtain an infinite set of Levy equations for the various approximations to the stress components. Further, if each set of approximations is required to satisfy the equilibrium relations, we obtain an infinite set of second order partial differential equations. Each set consists of a system of two equations and involves an Airy stress function and an undetermined function. The first set of these equations (the zero approximation to the stress components) is the system associated with a perfectly plastic material [6]. The known boundary conditions are considered as belonging to the zeroth approximation; the boundary stresses for σ_x, σ_{xy} associated with the remaining approximations are assigned the value zero. It is shown that the various approximations to the stresses can be determined by a step-by-step method of computation. Further, a condition for a line of rupture [7] is ob-

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[1] The numbers in brackets refer to the bibliography.

[2] This equation is $\left[\frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial x^2}\right]^2 + 4\left[\frac{\partial^2 F}{\partial x\,\partial y}\right]^2 - 4\left[\frac{b}{2}\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) + k\right]^2 = 0.$

[3] See [1].

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tained. Finally, it is shown that in the special case where σ_{xy} vanishes along $x = 0$, the approximations for σ_{xy} converge.

2. **Formulation of the problem.** We consider a material satisfying the Coulomb yield condition and occupying the semi-plane $x > 0$. At any point of the region occupied by this material, the following stress components exist: normal stresses, σ_x and σ_y, acting parallel to the x and y-axis, respectively; a shear stress σ_{xy}. We denote the mean normal stress by σ where by definition

$$2\sigma = \sigma_x + \sigma_y. \tag{2.1}$$

Further, if σ_1, σ_2 ($\sigma_1 > \sigma_2$) denote the two principal normal stresses and γ denotes the angle between the x-axis and the tangent to one family of lines of maximum shearing stress, then the Levy relations[4] are

$$\sigma_x = \sigma + \frac{\sigma_1 - \sigma_2}{2} \sin 2\gamma, \tag{2.2}$$

$$\sigma_y = \sigma - \frac{\sigma_1 - \sigma_2}{2} \sin 2\gamma, \tag{2.3}$$

$$\sigma_{xy} = -\frac{\sigma_1 - \sigma_2}{2} \cos 2\gamma. \tag{2.4}$$

The Coulomb yield condition may be written as

$$\frac{\sigma_1 - \sigma_2}{2} = b\sigma + k, \qquad k \neq 0, \tag{2.5}$$

where b and k are constants, the former having the nature of a friction coefficient.[5] We shall assume that the stresses σ_x, σ_y, σ_{xy} and the functions $\sin 2\gamma$, $\cos 2\gamma$ can be expanded in convergent power series of this friction coefficient and various unknown approximations (functions of x and y). That is,

$$\sigma_x = \sigma_x^{(0)} + b\sigma_x^{(1)} + b^2\sigma_x^{(2)} + \cdots b^n\sigma_x^{(n)} + \cdots, \tag{2.6}$$

$$\sigma_y = \sigma_y^{(0)} + b\sigma_y^{(1)} + b^2\sigma_y^{(2)} + \cdots b^n\sigma_y^{(n)} + \cdots, \tag{2.7}$$

$$\sigma_{xy} = \sigma_{xy}^{(0)} + b\sigma_{xy}^{(1)} + b^2\sigma_{xy}^{(2)} + \cdots b^n\sigma_{xy}^{(n)} + \cdots, \tag{2.8}$$

$$\sin 2\gamma = s^{(0)} + bs^{(1)} + b^2s^{(2)} + \cdots b^n s^{(n)} + \cdots, \tag{2.9}$$

$$\cos 2\gamma = c^{(0)} + bc^{(1)} + b^2c^{(2)} + \cdots b^n c^{(n)} + \cdots. \tag{2.10}$$

In view of the relation

$$\sin^2 2\gamma + \cos^2 2\gamma = 1, \tag{2.11}$$

the equations (2.9), (2.10) furnish relations between the approximations $c^{(n)}$ and $s^{(n)}$. These relations are of the form

$$[c^{(0)}]^2 + [s^{(0)}]^2 = 1, \tag{2.12}$$

[4] Relations (2.1), (2.2), (2.3) are dependent.

[5] See [1]

$$(2.13)\quad [c^{(n)}c^{(0)} + c^{(n-1)}c^{(1)} + \cdots c^{(0)}c^{(n)}] + [s^{(n)}s^{(0)} + s^{(n-1)}s^{(1)} + \cdots s^{(0)}s^{(n)}] = 0,\ n = 0, 1, 2, \cdots.$$

Further, by substituting (2.5) through (2.10) into equations (2.2) through (2.4), we obtain

$$(2.14)\quad \sigma_x^{(0)} = \sigma^{(0)} + ks^{(0)},$$

$$(2.15)\quad \sigma_y^{(0)} = \sigma^{(0)} - ks^{(0)},$$

$$(2.16)\quad \sigma_{xy}^{(0)} = -kc^{(0)},$$

$$(2.17)\quad \sigma_x^{(n)} = \sigma^{(n)} + ks^{(n)} + \sigma^{(0)}s^{(n-1)} + \sigma^{(1)}s^{(n-2)} + \cdots \sigma^{(n-1)}s^{(0)},$$

$$(2.18)\quad \sigma_y^{(n)} = \sigma^{(n)} - ks^{(n)} - \sigma^{(0)}s^{(n-1)} - \sigma^{(1)}s^{(n-2)} - \cdots \sigma^{(n-1)}s^{(0)},$$

$$(2.19)\quad \sigma_{xy}^{(n)} = -kc^{(n)} - \sigma^{(0)}c^{(n-1)} - \sigma^{(1)}c^{(n-2)} - \cdots \sigma^{(n-1)}c^{(0)},\ n = 1, 2, \cdots.$$

where by definition

$$(2.20)\quad 2\sigma^{(n)} = \sigma_x^{(n)} + \sigma_y^{(n)},\qquad n = 0, 1, 2, \cdots.$$

Because of the equation (2.20), it is easily seen that equations (2.14), (2.15) and (2.17), (2.18) are dependent, respectively. If we condition the approximations in (2.6), (2.7), (2.8) by requiring that σ_z, σ_y, σ_{xy}, satisfy the equilibrium relations[6] for all values of b, then we may introduce the Airy stress functions $F^{(n)}(x, y)$ such that

$$(2.21)\quad \sigma_x^{(n)} = \frac{\partial^2 F^{(n)}}{\partial y^2},\quad \sigma_y^{(n)} = \frac{\partial^2 F^{(n)}}{\partial x^2},\quad \sigma_{xy}^{(n)} = -\frac{\partial^2 F^{(n)}}{\partial x \partial y},\quad n = 0, 1, 2, \cdots.$$

Using (2.21) and subtracting equation (2.15) from (2.14), and equation (2.18) from (2.17), we obtain the systems

$$(2.22)\quad \frac{\partial^2 F^{(0)}}{\partial y^2} - \frac{\partial^2 F^{(0)}}{\partial x^2} = 2ks^{(0)},$$

$$(2.23)\quad \frac{\partial^2 F^{(0)}}{\partial x\, \partial y} = -kc^{(0)},$$

$$(2.24)\quad \frac{\partial^2 F^{(n)}}{\partial y^2} - \frac{\partial^2 F^{(n)}}{\partial x^2} = 2[ks^{(n)} + \sigma^{(0)}s^{(n-1)} + \sigma^{(1)}s^{(n-2)} + \cdots \sigma^{(n-1)}s^{(0)}],$$

$$(2.25)\quad \frac{\partial^2 F^{(n)}}{\partial x\, \partial y} = -[kc^{(n)} + \sigma^{(0)}c^{(n-1)} + \sigma^{(1)}c^{(n-2)} + \cdots \sigma^{(n-1)}c^{(0)}].$$

Note, the equations (2.22), (2.23) are the relations which determine the stress function for a perfectly plastic material.[7]

Our analysis of the boundary conditions will be similar to that used in [6]. Let the known boundary stresses be

$$(2.26)\quad \sigma_z(o, y) = \phi(y),\qquad \sigma_{xy}(o, y) = \theta(y),$$

[6] See [1]

[7] See [6], equations (2.6), (2.7).

where $\phi(y)$ and $\theta(y)$ are arbitrary functions of y which possess piecewise continuous second derivatives. We assume that the boundary conditions for the various approximations to the stresses are

$$\sigma_x^{(0)}(o, y) = \phi(y), \qquad \sigma_{xy}^{(0)}(o, y) = \theta(y), \tag{2.27}$$

$$\sigma_x^{(n)}(o, y) = 0, \qquad \sigma_{xy}^{(n)}(o, y) = 0, \; n = 1, 2, \cdots. \tag{2.28}$$

Hence, from (2.14), (2.16), (2.17), (2.19) and the above equation, we obtain the boundary conditions

$$kc^{(0)}]_{o,y} = \theta(y), \tag{2.29}$$

$$\sigma^{(0)} + ks^{(0)}]_{o,y} = \phi(y), \tag{2.30}$$

$$kc^{(n)} + \sigma^{(0)}c^{(n-1)} + \cdots \sigma^{(n-1)}c^{(0)}]_{o,y} = 0, \tag{2.31}$$

$$\sigma^{(n)} + ks^{(n)} + \sigma^{(0)}s^{(n-1)} + \cdots \sigma^{(n-1)}s^{(0)}]_{o,y} = 0, \; n = 1, 2, \cdots. \tag{2.32}$$

Equations (2.29) through (2.32) when used in conjunction with (2.12), (2.13) furnish boundary data for $\sigma^{(n)}$, $c^{(n)}$, $s^{(n)}$. Thus, (2.29) determines $c^{(0)}$; then (2.12) determines $s^{(0)}$; (2.30) determines $\sigma^{(0)}$; (2.31) determines $c^{(1)}$; (2.13) determines $s^{(1)}$; (2.32) determines $\sigma^{(1)}$, etc. From (2.12), we see that $s^{(0)}$ is undetermined to within a sign. The effect of this ambiguity is to interchange x and y wherever these variables occur. If the value of $\sigma^{(0)}$ relative to $\sigma_x^{(0)}$ is known, then by use of (2.30), the sign of $s^{(0)}$ may be determined. However, the above outlined step-by-step process does fail if $\theta(y) = \pm k$. In this special case, a simple computation shows that the relations (2.29) through (2.32) will be consistent only if $\phi(y)$ vanishes. Furthermore, the coefficients $s^{(n)}$ and $c^{(n)}$ are no longer uniquely determined. Hence, in our future work, we shall exclude those boundary points for which $\theta(y) = \pm k$.

By use of (2.21), (2.27), (2.28), we find the Cauchy data necessary for the integration of the systems (2.22) through (2.25). Thus, from (2.21), (2.27), we find by integration

$$F^{(0)}(o, y) = \alpha(y) = \int_{y_0}^{y} dy \int_{y_0}^{y} \phi(y)\, dy, \tag{2.33}$$

$$-\frac{\partial F^{(0)}}{\partial x}\Big]_{o,y} = \beta(y) = \int_{y_0}^{y} \theta(y)\, dy. \tag{2.34}$$

Similarly, by use of (2.21) and (2.28), we find

$$F^{(n)}(o, y) = 0, \qquad \frac{\partial F^{(n)}}{\partial x}\Big]_{o,y} = 0, \qquad n = 1, 2, \cdots. \tag{2.35}$$

It is to be noted that $\alpha(y)$ and $\beta(y)$ are undetermined to within a linear function of y and a constant, respectively. However, the stresses $\sigma_x^{(0)}$, $\sigma_y^{(0)}$, $\sigma_{xy}^{(0)}$ are completely determined when $\phi(y)$, $\theta(y)$ are known, as we shall show in the next section.

3. Integration. Our method of integration is similar to that used in [6].[8] First, we consider the stress function $F^{(0)}(x, y)$. By integrating (2.22), we find

$$(3.1)\quad F^{(0)}(\xi, \eta) = \frac{1}{2}\left[\alpha(\eta - \xi) + \alpha(\eta + \xi) + \int_{\eta+\xi}^{\eta-\xi} \beta(y)\, dy\right] - k \iint_{\triangle} s^{(0)}\, dx\, dy.$$

In (3.1), the coordinates (ξ, η) are those of an arbitrary point P in the plastic region R; the multiple integral is taken over the triangle bounded by the y-axis and the characteristics of (2.22) passing through the point P. Note, these characteristics are straight lines inclined at 45° and 135° to the x-axis.

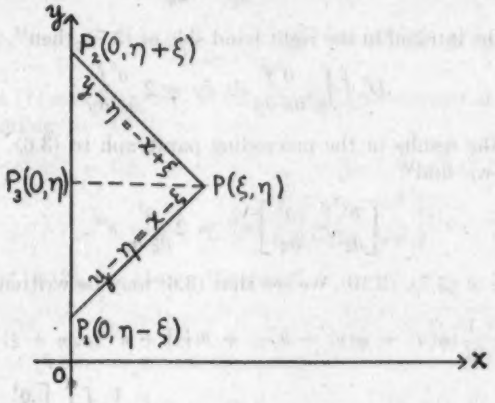

FIG. 1

By use of (2.21) and (3.1), we can calculate the zeroth approximation to the stresses in terms of the known boundary data and the unknown function $s^{(0)}$. We write

$$(3.2)\qquad u = \eta - \xi, \qquad v = \eta + \xi.$$

Forming the second derivatives of (3.1), we find

$$(3.3)\quad \sigma_\xi^{(0)} = \frac{\partial^2 F^{(0)}}{\partial \eta^2} = \tfrac{1}{2}[\phi(u) + \phi(v) + \theta(u) - \theta(v)] - k \frac{\partial^2}{\partial \eta^2} \iint_{\triangle} s^{(0)}\, dx\, dy,$$

$$(3.4)\quad \sigma_\eta^{(0)} = \frac{\partial^2 F^{(0)}}{\partial \xi^2} = \tfrac{1}{2}[\phi(u) + \phi(v) + \theta(u) - \theta(v)] - k \frac{\partial^2}{\partial \xi^2} \iint_{\triangle} s^{(0)}\, dx\, dy,$$

$$(3.5)\quad \sigma_{\xi\eta}^{(0)} = -\frac{\partial^2 F^{(0)}}{\partial \xi\, \partial \eta} = \tfrac{1}{2}[\phi(u) - \phi(v) + \theta(u) + \theta(v)] + k \frac{\partial^2}{\partial \xi\, \partial \eta} \iint_{\triangle} s^{(0)}\, dx\, dy.$$

Substituting (3.5) into (2.23), we obtain the integral-differential relation between $s^{(0)}$ and $c^{(0)}$,

$$(3.6)\qquad c^{(0)} = \frac{1}{2k}[\phi(u) - \phi(v) + \theta(u) + \theta(v)] + \frac{\partial^2}{\partial \xi\, \partial \eta} \iint_{\triangle} s^{(0)}\, dx\, dy.$$

[8] See [6], section 3.

Equation (3.6) has been solved in [6]. Here, we shall briefly review the highlights of our former results in a form applicable to finding $c^{(n)}$. If $f(x, y)$ possesses continuous second partial derivatives in the domain considered, then it can be shown that[9]

$$\frac{\partial^2}{\partial\xi\,\partial\eta}\iint_{\triangle} f(x, y)\,dx\,dy = \iint_{\triangle} \frac{\partial^2 f}{\partial x\,\partial y}\,dx\,dy + f(o, \eta + \xi) - f(o, \eta - \xi). \tag{3.7}$$

Further, it can be shown that if the operator

$$D^2 = \frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\eta^2}, \tag{3.8}$$

is applied to the integral in the right hand side of (3.7), then[10]

$$D^2 \iint_{\triangle} \frac{\partial^2 f}{\partial x\,\partial y}\,dx\,dy = 2\frac{\partial^2 f}{\partial x\,\partial y}. \tag{3.9}$$

We apply the results of the preceeding paragraph to (3.6). Operating with D^2 on (3.6), we find[11]

$$\left[\frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\eta^2}\right]c^{(0)} = 2\frac{\partial^2}{\partial\xi\,\partial\eta}s^{(0)}. \tag{3.10}$$

Hence, by use of (3.7), (3.10), we see that (3.6) may be written as

$$\begin{aligned} c^{(0)} = {} & \frac{1}{2k}[\phi(u) - \phi(v) + \theta(u) + \theta(v)] + s^{(0)}(o, \eta + \xi) - s^{(0)}(o, \eta - \xi) \\ & + \frac{1}{2k}\iint_{\triangle}\left[\frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\eta^2}\right]c^{(0)}\,dx\,dy. \end{aligned} \tag{3.11}$$

In view of a characteristic property[11] of the operator D^2, it follows by direct substitution into (3.11) that

$$c^{(0)} = \frac{1}{2k}[\phi(u) - \phi(v) + \theta(u) + \theta(v)] + s^{(0)}(o, \eta + \xi) - s^{(0)}(o, \eta - \xi) \tag{3.12}$$

is the solution of (3.11). Further, by use of (2.29) and (2.12), the value of $s^{(0)}$ along the boundary may be computed in terms of known functions. Hence, $c^{(0)}$ is completely determined at P. It follows that $s^{(0)}$ at P may be determined by use of (2.12). The ambiguity of sign in $s^{(0)}$ may be removed by knowledge of the sign of $s^{(0)}$ along the boundary (see discussion following 2.32). Further, the value of $\sigma^{(0)}$ at P may be found by use of (3.3), (3.4).

Next, let us consider the first approximation. Integrating the equation (2.24) for $n = 1$ and using the proper boundary data (2.35), we find

$$F^{(1)}(\xi, \eta) = -\iint_{\triangle}[ks^{(1)} + \sigma^{(0)}s^{(0)}]\,dx\,dy. \tag{3.13}$$

[9] See [6], equation (3.20)

[10] See [6], equation (3.26).

[11] Note, if $g(v)$ and $f(u)$ possess continuous second derivatives then $D^2f(u) = D^2g(v) = 0$.

Hence, it follows that

$$\sigma_\xi^{(1)} = \frac{\partial^2 F^{(1)}}{\partial \eta^2} = -\frac{\partial^2}{\partial \eta^2}\iint_\triangleright [ks^{(1)} + \sigma^{(0)} s^{(0)}]\, dx\, dy, \tag{3.14}$$

$$\sigma_\eta^{(1)} = \frac{\partial^2 F^{(1)}}{\partial \xi^2} = -\frac{\partial^2}{\partial \xi^2}\iint_\triangleright [ks^{(1)} + \sigma^{(0)} s^{(0)}]\, dx\, dy, \tag{3.15}$$

$$\sigma_{\xi\eta}^{(1)} = -\frac{\partial^2 F^{(1)}}{\partial \xi\, \partial \eta} = \frac{\partial^2}{\partial \xi\, \partial \eta}\iint_\triangleright [ks^{(1)} + \sigma^{(0)} s^{(0)}]\, dx\, dy. \tag{3.16}$$

Further, by use of (3.16) and (2.25), we obtain

$$kc^{(1)} + \sigma^{(0)} c^{(0)} = \frac{\partial^2}{\partial \xi\, \partial \eta}\iint_\triangleright [ks^{(1)} + \sigma^{(0)} s^{(0)}]\, dx\, dy. \tag{3.17}$$

The equation (3.17) may be solved in exactly the same manner as equation (3.6). Thus, corresponding to (3.11), we have

$$\begin{aligned} kc^{(1)} + \sigma^{(0)} c^{(0)} &= [ks^{(1)} + \sigma^{(0)} s^{(0)}]_{o,\eta+\xi} - [ks^{(1)} + \sigma^{(0)} s^{(0)}]_{o,\eta-\xi} \\ &\quad + \frac{1}{2}\iint_\triangleright \left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right][kc^{(1)} + \sigma^{(0)} c^{(0)}]\, dx\, dy. \end{aligned} \tag{3.18}$$

Reasoning as in (3.12), we obtain the result

$$kc^{(1)} + \sigma^{(0)} c^{(0)} = [ks^{(1)} + \sigma^{(0)} s^{(0)}]_{o,\eta+\xi} - [ks^{(1)} + \sigma^{(0)} s^{(0)}]_{o,\eta-\xi}. \tag{3.19}$$

Finally, by use of (2.32), we obtain the result

$$kc^{(1)} + \sigma^{(0)} c^{(0)} = \sigma^{(1)}(o, \eta - \xi) - \sigma^{(1)}(o, \eta + \xi). \tag{3.20}$$

It should be noted that the known boundary value conditions furnish the right hand side of (3.20). Further, $c^{(0)}$ is determined by (3.12) and $\sigma^{(0)}$ is found by use of (3.3), (3.4). Hence, equation (3.20) furnishes $c^{(1)}$ at $P(\xi, \eta)$. The quantity $s^{(1)}$ may be computed by use of (2.13) and hence $\sigma^{(1)}$ may be found from (3.14), (3.15). This step-by-step method of calculating the approximations may be continued to $c^{(2)}$ etc. We shall list the results for the n^{th} approximation.

$$F^{(n)}(\xi, \eta) = -\iint_\triangleright [ks^{(n)} + \sigma^{(0)} s^{(n-1)} + \cdots \sigma^{(n-1)} s^{(0)}]\, dx\, dy, \tag{3.21}$$

$$\sigma_\xi^{(n)} = \frac{\partial^2 F^{(n)}}{\partial \eta^2} = -\frac{\partial^2}{\partial \eta^2}\iint_\triangleright [ks^{(n)} + \sigma^{(0)} s^{(n-1)} + \cdots \sigma^{(n-1)} s^{(0)}]\, dx\, dy, \tag{3.22}$$

$$\sigma_\eta^{(n)} = \frac{\partial^2 F^{(n)}}{\partial \xi^2} = -\frac{\partial^2}{\partial \xi^2}\iint_\triangleright [ks^{(n)} + \sigma^{(0)} s^{(n-1)} + \cdots \sigma^{(n-1)} s^{(0)}]\, dx\, dy, \tag{3.23}$$

$$\sigma_{\xi\eta}^{(n)} = -\frac{\partial^2 F^{(n)}}{\partial \xi\, \partial \eta} = \frac{\partial^2}{\partial \xi\, \partial \eta}\iint_\triangleright [ks^{(n)} + \sigma^{(0)} s^{(n-1)} + \cdots \sigma^{(n-1)} s^{(0)}]\, dx\, dy, \tag{3.24}$$

$$\begin{aligned} &kc^{(n)} + \sigma^{(0)} c^{(n-1)} + \cdots \sigma^{(n-1)} c^{(0)} = \sigma^{(n)}(o, \eta - \xi) - \sigma^{(n)}(o, \eta + \xi), \\ &\qquad \eta = 1, 2, \cdots. \end{aligned} \tag{3.25}$$

Further, from (2.25), (3.25), we see that

$$\sigma_{\xi\eta}^{(n)} = \sigma^{(n)}(o, \eta - \xi) - \sigma^{(n)}(o, \eta + \xi), \qquad n = 1, 2, \cdots, \tag{3.26}$$

and from (2.23), (3.12), we find that the corresponding equation for the zeroth approximation is

$$\sigma_{\xi\eta}^{(0)} = \tfrac{1}{2}[\phi(\eta - \xi) - \phi(\eta + \xi) + \theta(\eta - \xi) + \theta(\eta + \xi)] + ks^{(0)}(o, \eta + \xi) - ks^{(0)}(o, \eta - \xi). \tag{3.27}$$

These last two relations will be used in discussing the convergence of $\sigma_{\xi\eta}^{(n)}$.

It should be noted that the previously outlined method of step-by-step calculation fails if at $P(\xi, \eta)$

$$s^{(0)} = 0 \quad \text{or} \quad c^{(0)} = \pm 1. \tag{3.28}$$

Further, the arguments used in [6][12] show that as $P(\xi, \eta)$ approaches a point on the locus (3.28), the quantity $\partial\sigma_{\eta}^{(0)}/\partial\xi$ becomes unbounded. Following the terminology of S. Christianovich [7], we shall call the locus (3.28) a line of rupture. Thus, our method of determining the stresses fails at two types of points: (1) those points $P(\xi, \eta)$ for which the power series (2.6) through (2.10) fail to converge; (2) those points $P(\xi, \eta)$ which lie on the locus (3.28).

4. **Convergence of the approximations $\sigma_{\xi\eta}^{(n)}$.** We shall show that the approximations $\sigma_{\xi\eta}^{(n)}$ converge in the case

$$\sigma_{xy}(o, y) = \theta(y) = 0. \tag{4.1}$$

First, we consider the values of $\sigma^{(n)}$ along the boundary $x = 0$. From (4.1), (2.29), we conclude[13]

$$c^{(0)} = 0, \qquad s^{(0)} = 1. \tag{4.2}$$

By use of (2.31) and the first relation in (4.2), we obtain

$$c^{(n)} = 0, \qquad n = 1, 2, \cdots. \tag{4.3}$$

Substituting (4.2) into (2.13), we find

$$s^{(n)} = 0, \qquad n = 1, 2, \cdots. \tag{4.4}$$

Further, from (4.2), (4.4), and (2.32), we obtain

$$\sigma^{(n)} + \sigma^{(n-1)} = 0, \qquad \eta = 1, 2, \cdots. \tag{4.5}$$

Hence, the power series

$$\sigma = \sigma^{(0)} + b\sigma^{(1)} + b^2\sigma^{(2)} + \cdots b^n\sigma^{(n)} + \cdots, \tag{4.6}$$

for the mean normal stress σ along the boundary, converges if $|b| < 1$. It should be noted that only for such values of b do two distinct families of slip

[12] See [6], equations (3.30) through (3.35).

[13] We omit the possible value $s^{(0)} = -1$.

lines exist.[14] Finally, we note that $\sigma^{(n)}$ may be explicitly determined by use of (4.2), (4.5), and (2.30). We find

$$\sigma^{(0)} = \phi(y) - k, \qquad \sigma^{(n)} = (-1)^n[\phi(y) - k], \; n = 1, 2, \cdots. \tag{4.7}$$

Next, we consider the stresses $\sigma_{\xi\eta}^{(n)}$ acting at $P(\xi, \eta)$. From (4.7) and (3.26), we find

$$\sigma_{\xi\eta}^{(n)} = (-1)^n[\phi(\eta - \xi) - \phi(\eta + \xi)], \qquad n = 1, 2, \cdots. \tag{4.8}$$

Further, by use of (4.1), (4.2), and (3.27), we obtain

$$\sigma_{\xi\eta}^{(0)} = \tfrac{1}{2}[\phi(\eta - \xi) - \phi(\eta + \xi)]. \tag{4.9}$$

The relations (4.8) are sufficient to prove that the series (2.8) converges for $|b| < 1$. In fact, by use of (4.8), (4.9), the series (2.8) may be summed. We obtain

$$\sigma_{\alpha\beta} = \frac{1 - b}{2(1 + b)}[\phi(\eta - \xi) - \phi(\eta + \xi)]. \tag{4.10}$$

The discussion of the convergence of the stresses $\sigma_\xi^{(n)}$, $\sigma_\eta^{(n)}$ is difficult even in the case (4.1). The principal difficulty lies in estimating $\sigma^{(n)}$ from the equations (3.22), (3.23).

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[14] See [3], theorem 5.

TRANSVERSALITY IN HIGHER SPACE

By Jacques Dutka

Transversality relations first arose in problems of the calculus of variations. In deriving sufficiency conditions for a curve to exist which makes

(1) $I = \int_a^b f(x, y, y')\,dx$ a minimum, Weierstrass[1] introduced the concepts of field and the ϵ function which is closely related to the transversality condition:

$$f(x, y, p) + (\bar{y}' - p)f_{y'}, (x, y, p) = 0 \tag{2}$$

that must be satisfied when one of the limits of integration is permitted to vary along $\bar{y} = \bar{y}(x)$. Hilbert[2] used this function in finding other sufficiency conditions and defined the integral, since named after him which we shall write as

$$H(x, y) = \int_{(a,y(a))}^{(x,y)} \{(f(x, y, p) - pf_{y'}(x, y, p)\,dx + f_{y'}(x, y, p)\,dy\} \tag{3}$$

The curves $H(x, y) =$ constant are called the transversals. Denoting the slope $\frac{dy}{dx}$ of the transversals by $\bar{y}'$ and substituting in the differential equation of the family H, we obtain (2). At every point (x, y) the direction $\bar{y}'$ is transversal to the direction $p(x, y)$ of the extremal passing through that point. The term transversal was introduced by Kneser[3] and used in formulating his fundamental theorem.

Transversality relations also came up in Lie's theory of infinitesimal contact transformations but were not explicitly named such. Kasner[4] first proposed the use of transversal in this connection.

A transversality in n-space may be defined as a special one-to-one correspondence between lineal elements and hypersurface elements having a common base point. It is the object of this paper to investigate this correspondence more closely. Vessiot[5] considered these correspondences in connection with

[1] See Schwarz's *Abhandlungen*, vol. 1, p. 223. A field is defined as a one parameter family of extremals $y = y(x, a)$ of (1) such that through every point of a region there passes exactly one extremal.

$$E(x, y, \bar{y}') = f(x, y, \bar{y}') - f(x, y, p) - (\bar{y}' - p)f_{y'}(x, y, p)$$

where $p = p(x, y)$ is the slope function and $\bar{y}$ is a comparison curve lying entirely in the field.

[2] See *Bericht des Mathematischen Vereins*, Göttingen, 1900, p. 10.

[3] Kneser, *Lehrbuch der Variationsrechnung*, Braunschweig, 1900, p. 57 cf, however Bolza, *Vorelesungen über Variationsrechnung*, Leipzig, p. 44, 1909, Note 2.

Kneser's theorem asserts that two transversals $H(x, y) = c_1$ and $H(x, y) = c_2$ of a field intersecting the extremals cut off arcs on the extremals for which the integral $\int f(x, y, y')\,dx$ has the same value.

[4] *The infinitesimal contact transformations of mechanics*, Bull. A. M. S., vol. 16 (1910), pp. 408–412.

[5] Vessiot, *Sur L'Interpretation Mecanique Des Transformations de Contact Infinitesimales*, Bulletin de la Societe Mathematique de France, vol. 34 (1906), pp. 230–269.

infinitesimal transformations and their relations to a class of problems of the calculus of variations in three dimensions. Douglas[6] developed this method further in a general theorem characterizing the composite of extremals and transversality which is valid in n-dimensions. Kasner[7] found a fundamental geometric criterion for a transversality to belong to a three dimensional problem of the calculus of variations. It is here shown that in n-dimensions, a necessary and sufficient condition for a given correspondence between a lineal element and a hypersurface element to be a transversality is that a certain induced correlation should be a polarity.

In this paper, the results are developed in space of four dimensions for brevity of notation and it is finally shown how they can be carried over to n-dimensions *mutatis mutandis*.

Let us consider the infinitesimal contact transformation defined by the characteristic function

$$W(x, y, z, v, p, q, r), \qquad p = \frac{\partial v}{\partial x}, \qquad q = \frac{\partial v}{\partial y}, \qquad r = \frac{\partial v}{\partial z} \tag{4}$$

the simple integral

$$\int G(x, y, z, v, y', z', v')\, dx = \text{minimum} \tag{5}$$

and the 3-fold integral

$$\iiint F(x, y, z, v, p, q, r)\, dx\, dy\, dz = \text{minimum} \tag{6}$$

In (4) the infinitesimal contact transformation carries any hypersurface element (x, y, z, v, p, q, r) into a nearby element $(x + \delta x, \cdots, r + \delta r)$. The equations of transformation are given by

$$\delta x = W_p \delta t,\ \delta y = W_q \delta t,\ \delta z = W_r \delta t, \qquad \delta v = (pW_p + qW_q + rW_r - W)\delta t \tag{7}$$

disregarding terms $O(\delta t^2)$. Connecting the base point of the first element with that of the second, we obtain a definite direction element, i.e. (x, y, z, v, y', z', v') where

$$y' = \frac{W_q}{W_p}, \qquad z' = \frac{W_r}{W_p}, \qquad v' = \frac{pW_p + qW_q + rW_r - W}{W_p} \tag{8}$$

The lineal element is transversal to the hypersurface element. That is, fixing the base point (x, y, z, v), there is a definite association or correspondence between the ∞^3 direction elements (y', z', v') and the ∞^2 tangent three-flats[8] $(p,$

[6] Douglas, *Extremals and Transversality*. . . , Trans. Amer. Math. Soc., vol. 29 (1927), pp. 401–420.

[7] Kasner, *Transversality in Space of Three Dimensions*, Trans. Amer. Math. Soc., vol. 30 (1928), pp. 447–452. Professor Kasner first called that author's attention to the possibility of extending these results to higher dimensions.

[8] The terminology has been adopted from Forsyth's two volume work, *Geometry of Four Dimensions*, Cambridge Press (1930).

q, r). It will be seen later that when the direction elements are given the hypersurface elements are also fully determined so that the correspondence is biunique.

In problem (5) the end point of a curve minimizing the integral is constrained to lie on a given hypersurface $\varphi(x, y, z, v) = 0$. The transversality of the tangent three-flat (p, q, r) of φ to the direction element (y', z', v') of the curve at that point is given by the relationship

$$(9) \qquad (y'G_{y'} + z'G_{z'} + v'G_{v'} - G)\delta x - G_{y'}\delta y - G_{z'}\delta z - G_{v'}\delta v = 0$$

or

$$p = \frac{y'G_{y'} + z'G_{z'} + v'G_{v'} - G}{G_{v'}}, \qquad q = -\frac{G_{y'}}{G_{v'}}, \qquad r = -\frac{G_{z'}}{G_{v'}}$$

This is readily recognized to be a dual form of (8), in which the hypersurface elements are determined when the direction elements are given. Making the transformations

$$(10) \qquad P = -v', Q = y', R = z', Y' = q, Z' = r, V' = p, \bar{G}(P, Q, R) = G(Q, R, -P)$$

we obtain

$$(11) \qquad Y' = \frac{\bar{G}_Q}{\bar{G}_P}, \qquad Z' = \frac{\bar{G}_R}{\bar{G}_P}, \qquad V' = \frac{P\bar{G}_P + Q\bar{G}_Q + R\bar{G}_R - \bar{G}}{\bar{G}_P}$$

which is of the same form as (8) a correspondence from the hypersurface elements (P, Q, R) to the line elements (Y', Z', V').

In problem (6), a hypersurface $\psi(x, y, z, v) = 0$ is said to be transversal to a given field $\mathfrak{F}$ of extremals if ψ satisfies the partial differential equation

$$(12) \qquad F_p\psi_x + F_q\psi_y + F_r\psi_z + (pF_p + qF_q + rF_r - F)\psi_v = 0$$

Analogously to (2) and (3), on every portion of one of these hypersurfaces, the Hilbert Integral vanishes. The set of transversal curves of the field $\mathfrak{F}$ are the characteristic curves of the equation (12). They satisfy the ordinary differential equations

$$(13) \qquad \frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{F_r} = \frac{dv}{pF_p + qF_q + rF_r - F}$$

Computing the directions y', z', v' we have the same form as (8) and (11). At a given point ψ goes through a certain direction determined by and transversal to the tangent three-flat element of the extremal surface at that point.

Given any one function say W in (4), the other integrand functions F and G in (5) and (6) are determined up to a factor of the form $\lambda(x, y, z, v)$ and similarly for F and G. It is known that transversality is unchanged by such a factor in any one of the problems (4), (5) and (6). Thus we have established a principle of transference associating contact transformations, simple and three-fold integrals in four dimensional space when they determine equivalent

transversalities. The three kinds of transversalities reduced to one. Therefore in order to study the transversalities of problems (4), (5), (6) we have only to turn our attention to one of them, the results are applicable to all.

A correspondence of the form (8) is of special analytic character. There is only one arbitrary function W depending on seven variables whereas a general analytic correspondence would be of the form

$$(14) \qquad T: \quad y' = \alpha(x, y, z, v, p, q, r), \quad z' = \beta(x, y, z, v, p, q, r),$$
$$v' = \gamma(x, y, z, v, p, q, r), \qquad J \neq 0$$

namely three random functions each involving seven arguments. To formulate an analytic criterion for transversality we rewrite the equations (8) in the form

$$(15) \qquad \alpha = \frac{W_q}{W_p}, \qquad \beta = \frac{W_r}{W_p}, \qquad \gamma = \frac{pW_p + qW_q + rW_r - W}{W_p}$$

To obtain a relation independent of W, we form the equations

$$(16) \qquad \frac{W_p}{W} = \frac{1}{p + q\alpha + r\beta - \gamma}, \qquad \frac{W_q}{W} = \frac{\alpha}{p + q\alpha + r\beta - \gamma},$$
$$\frac{W_r}{W} = \frac{\beta}{p + q\alpha + r\beta - \gamma}$$

from (15). Eliminating W by setting up the $\binom{3}{2} = 3$ conditions for integrability, i.e.

$$(17) \qquad \text{a) } \frac{\partial}{\partial q}\frac{W_p}{W} = \frac{\partial}{\partial p}\frac{W_q}{W} \qquad \text{b) } \frac{\partial}{\partial r}\frac{W_p}{W} = \frac{\partial}{\partial p}\frac{W_r}{W} \qquad \text{c) } \frac{\partial}{\partial q}\frac{W_r}{W} = \frac{\partial}{\partial r}\frac{W_q}{W}$$

we obtain by differentiating (16) partially

$$(18) \quad \text{a) } q\alpha_q + r\beta_q - \gamma_q + (p + r\beta - \gamma)\alpha_p - r\alpha\beta_p + \alpha\gamma_p = 0$$
$$\text{b) } q\alpha_r + r\beta_r - \gamma_r + (p + q\alpha - \gamma)\beta_p - q\beta\alpha_p + \beta\gamma_p = 0$$
$$\text{c) } (p + q\alpha - \gamma)\beta_q - \beta(q\alpha_q - \gamma_q) - (p + r\beta - \gamma)\alpha_r + \alpha(r\beta_r - \gamma_r) = 0$$

This is the criterion for transversality in analytic form. It is both necessary and sufficient. If in (9) we had set up integrability conditions and eliminated G we would have obtained three conditions which are the duals of (18).

To obtain a geometric criterion equivalent to the analytic conditions that a transversality belongs to an infinitesimal contact transformation (4) or to problems of the calculus of variations (5) and (6), consider the transformation T associating hypersurface and lineal elements. T is assumed to be non-singular. At an arbitrary fixed point O we direct our attention to the three flat (p, q, r) denoted by π_1, through it.

There is induced by T a correspondence between the directions in π_1 at the point O and the planes in π_1 passing through O. This correspondence is defined as follows:

Let π_2 be the nearby three flat $(p + dp, q + dq, r + dr)$. The directions

associated with π_1 and π_2 by T namely $l_1 : (\alpha, \beta, \gamma)$ and $l_2 : (\alpha + d\alpha, \beta + d\beta, \gamma + d\gamma)$ span a plane Q. Let the line L be the intersection of Q with π_1. Further let P be the plane of cleavage of π_1 and π_2.

Evidently L and P depend on the choice of π_2, that is, on dp, dq, dr. We shall see that

1) the lines $\{L\}$ obtained by varying dp, dq, dr, arbitrarily, include all lines in π_1 through O;

2) the correspondence L to P is independent of the choice of dp, dq, dr i.e., if L and P are associated with dp, dq, dr and L' and P' are associated with dp', dq', dr', then the relation $L \equiv L'$ implies that $P \equiv P'$;

3) the correspondence L to P is linear, non-singular hence a (proper) correlation.

Consider an arbitrary correspondence

$$(14') \quad T: \quad y' = \alpha(p, q, r), \quad z' = \beta(p, q, r), \quad v' = \gamma(p, q, r), \quad J \neq 0$$

between the hypersurface elements and the direction elements at a given point O regarded as fixed. Since the relation of transversality is not affected by a rigid motion of the coordinate system, the relations (18) retain their form. Accordingly we may choose O as the origin and the flat π_1 through it, regarded as fixed, as the x, y, z coordinate space. Corresponding to π_1 whose equation is $v = 0$, there is a direction $l_1 : \ 1 : \alpha_0 : \beta_0 : \gamma_0$. Here $\alpha_0 = \alpha(0, 0, 0, -1)$ where $(0, 0, 0, -1)$ are the direction numbers of π_1 and similarly for β_0 and γ_0. To a consecutive three flat element π_2 denoted by $xdp + ydq + rdz - dv = 0$, corresponds the direction $l_2 : \ 1 : \alpha_0 + d\alpha : \beta_0 + d\beta : \gamma_0 + d\gamma$. Here $d\alpha = \alpha_p dp + \alpha_q dq + \alpha_r dr$ and $d\beta, d\gamma$ have the same form. These directions l_1 and l_2 determine a plane Q. Vector $l_2 - l_1$ is represented by the equations

$$(19) \quad l_2 - l_1 : \quad x = 0t + 1, \quad y = td\alpha + \alpha_0, \quad z = td\beta + \beta_0, \quad v = td\gamma + \gamma_0$$

To find the intersection of Q with π_1 namely L, we put $v = 0$ and from (19) we get $t = -\dfrac{\gamma_0}{d\gamma}$. The equations of the line L, therefore become

$$(20) \qquad L: \quad \frac{x}{-d\gamma} = \frac{y}{\gamma_0 d\alpha - \alpha_0 d\gamma} = \frac{z}{\gamma_0 d\beta - \beta_0 d\gamma}$$

π_1 and π_2 intersect in a plane of cleavage P whose equations are

$$(21) \qquad xdp + ydq + zdr = 0, \qquad v = 0$$

For the proof of 1) it will suffice to show that the functional matrix M of the direction numbers of L with respect to dp, dq, dr is of rank three. M may be written as

$$\begin{Vmatrix} -\gamma_p & -\gamma_q & -\gamma_r \\ (\gamma_0\alpha_p - \alpha_0\gamma_p) & (\gamma_0\alpha_q - \alpha_0\gamma_q) & (\gamma_0\alpha_r - \alpha_0\gamma_r) \\ (\gamma_0\beta_p - \beta_0\gamma_p) & (\gamma_0\beta_q - \beta_0\gamma_q) & (\gamma_0\beta_r - \beta_0\gamma_r) \end{Vmatrix}$$

As is well known, the rank of the matrix is left invariant under elementary transformations. Consequently M takes the form

$$\bar{M}: \begin{Vmatrix} -\gamma_p & -\gamma_q & -\gamma_r \\ \gamma_0\alpha_p & \gamma_0\alpha_q & \gamma_0\alpha_r \\ \gamma_0\beta_p & \gamma_0\beta_q & \gamma_0\beta_r \end{Vmatrix}$$

The determinant of this matrix is non-vanishing since it is the product of the non-zero Jacobian of (14′) and the factor γ_0^3 which may be taken as non-zero. (In the contrary case, for $\gamma_0 = 0$, equations (16) are meaningless. Geometrically however, l_1, is perpendicular to π_1. The transversality relation in this case is simply orthogonality—the characteristic property of natural families.) It follows therefore that M is of rank three and thus 1) is proved.

For the proof of 2) denote the direction numbers of L depending on dp, dq, dr by $\bar{x}$, $\bar{y}$, $\bar{z}$:

$$\begin{aligned} \bar{x} &= -(\gamma_p\,dp + \gamma_q\,dq + \gamma_r\,dr) \\ \bar{y} &= (\gamma_0\alpha_p - \alpha_0\gamma_p)\,dp + (\gamma_0\alpha_q - \alpha_0\gamma_q)\,dq + (\gamma_0\alpha_r - \alpha_0\gamma_r)\,dr \\ \bar{z} &= (\gamma_0\beta_p - \beta_0\gamma_p)\,dp + (\gamma_0\beta_q - \beta_0\gamma_q)\,dq + (\gamma_0\beta_r - \beta_0\gamma_r)\,dr \end{aligned} \tag{22}$$

The corresponding direction numbers of L' depending on dp', dq', dr' are denoted by $\bar{x}'$, $\bar{y}'$, $\bar{z}'$, respectively. If $L \equiv L'$, then the direction numbers of L are respectively proportional to the corresponding direction numbers of L'. We thus obtain three equations in the three quantities $(dp - kdp')$, $(dq - kdq')$, $(dr - kdr')$ where k is the constant of proportionality. The right hand members of the equations are zero.

We shall see that these quantities are each equal to zero and that consequently the planes associated with L and L', namely P and P' are equivalent. The matrix of the coefficients of these quantities is M or the equivalent form $\bar{M}$. For the homogeneous equations to have non-zero solutions, the determinant of the coefficients must vanish. But this is contradicted by the non-vanishing of the Jacobian of (14′). Consequently the quantities $(dp - kdp')$, $(dq - kdq')$, $(dr - kdr')$ are zero identically and $P \equiv P'$.

For the proof of 3), dp, dq, dr, denote the direction numbers of the plane P. The direction numbers of L: $\bar{x}$, $\bar{y}$, $\bar{z}$, given by equations (22) are linear in dp, dq, dr. The matrix of coefficients of these numbers namely M is non-singular, hence the correspondence between L and P is a (proper) correlation. If the matrix is symmetric, this induced correlation is a polarity and conversely. *A fortiori* we have then

$$\begin{aligned} -\gamma_q - \gamma_0\alpha_p + \alpha_0\gamma_p &= 0 \\ -\gamma_r - \gamma_0\beta_p + \beta_0\gamma_p &= 0 \\ -\gamma_0\beta_q + \beta_0\gamma_q + \gamma_0\alpha_r - \alpha_0\gamma_r &= 0 \end{aligned} \tag{23}$$

But these relations are simply what (18) becomes when p, q, r are set equal to zero. Since the conditions (18) were necessary and sufficient for transver-

sality, the criterion that the induced correlation between the plane P and the line L be a polarity is also *necessary and sufficient.*

In three dimensions the correlation reduces to a homography and the condition for transversality is that the homography be involutorial. This is precisely the condition obtained by Kasner who pointed out that if the correspondence between the three-flat and the line l_1 is assumed to be projective, it will satisfy the transversality condition when the correspondence is reciprocity with respect to any quadric hypercone at the given point. An analogous statement holds for n dimensions where $n \geq 3$. To put it differently, hypersurface transversality is projective only when it reduces to duality.

The results proved here are valid in n-dimensional homaloidal space but the calculations are much lengthier. The infinitesimal contact transformation

$$W(x_1, x_2, \cdots, x_n, p_1, p_2, \cdots, p_{n-1}), \qquad p_i = \frac{\partial x_n}{\partial x_i}, \qquad i = 1, 2, \cdots, n-1$$

the simple integral

$$\int G(x_1, x_2, \cdots, x_n, x_2', x_3', \cdots, x_n')\, dx_1 = \text{minimum},$$

$$x_i' = \frac{dx_i}{dx_1}, \qquad i = 2, 3, \cdots, n$$

and the $(n-1)$ fold integral

$$\iint \cdots \int F(x_1, x_2, \cdots, x_n, p_1, p_2, \cdots, p_{n-1})\, dx_1\, dx_2 \cdots dx_{n-1} = \text{minimum}$$

have equivalent transversalities. There are $(n-1)$ equations of the form (8) and eliminating W we obtain $\binom{n-1}{2}$ integrability conditions from the cross derivatives. The geometric reasoning carries through in an analogous manner and we finally obtain a bilinear form in $\delta p_1, \delta p_2, \cdots, \delta p_{n-1}$ and $dp_1, dp_2, \cdots, dp_{n-1}$ containing $(n-1)^2$ elements. The symmetry of this form, necessary for polarity, yields $(n-1)^2 - \binom{n}{2} = \binom{n-1}{2}$ conditions which are identical with the $\binom{n-1}{2}$ integrability conditions previously obtained. Here too, the necessary and sufficient condition for transversality is that a certain induced correlation is a polarity.

In two dimensions transversality relations are of arbitrary character. In the simplest problem of the calculus of variations, the integrand has a transversality given by

$$f(x, y, p) + (\bar{y}' - p) f_{y'}(x, y, p) = 0 \tag{2}$$

Rewriting the equation in the form $\frac{f_p}{f} = \frac{1}{p - \bar{y}'}$ and solving this differential

equation we have $f = g(x, y)e^{\theta}$ where $\theta = \int_0^p \frac{dt}{t - \bar{y}'}$. Stromquist[9] has given some examples where $\bar{y}' = \frac{ap + b}{cp + d}$, $\Delta \neq 0$. In particular the cases where the transversals are isogonal and orthogonal to the extremals are discussed.

In spaces of dimensionality $n \geq 3$, only special correspondences can be transversalities. In three dimensions if we consider the characteristic function $W(x, y, z, p, q)$, $p = z_x$, $q = z_y$, and set up the equations analogous to (16) namely

$$(24) \qquad \frac{W_p}{W} = \frac{1}{p + q\alpha - \beta}, \qquad \frac{W_q}{W} = \frac{\alpha}{p + q\alpha - \beta}, \qquad \begin{matrix} y' = \alpha(p, q) \\ z' = \beta(p, q) \end{matrix}$$

and solve, we find the solutions

$$\log W = \int_0^p \frac{dt}{t + q\alpha - \beta} + \log F_1(x, y, z, q);$$

$$\log W = \int_0^q \frac{\alpha\, dt}{p + t\alpha - \beta} + \log F_2(x, y, z, p)$$

thus

$$(25) \qquad W = F_1 \exp\left(\int_0^p \frac{dt}{t + q\alpha - \beta}\right) + F_2 \exp\left(\int_0^q \frac{\alpha\, dt}{p + t\alpha - \beta}\right)$$

If we form the integrability conditions from (24) and set the equation equal to zero to obtain particular solutions we get

$$(26) \qquad \frac{\partial}{\partial q}\frac{1}{p + q\alpha - \beta} = \frac{\partial}{\partial p}\frac{\alpha}{p + q\alpha - \beta} = 0$$

Solving (26)

$$(27) \qquad \frac{1}{p + q\alpha - \beta} = A(p), \qquad \frac{\alpha}{p + q\alpha - \beta} = B(q)$$

thus

$$(28) \qquad \alpha = \frac{B}{A}, \qquad \beta = \frac{pA + qB - 1}{A}$$

The Jacobian $J = \frac{1}{A^3}\begin{vmatrix} -BA_p & A^2 + A_p B \\ B_q & B \end{vmatrix}$ is non-vanishing and thus (28) is non-trivial.

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[9] Stromquist, *Annals of Mathematics*, (2), vol. 9 (1907) p. 57.

ON THE FOUNDATIONS OF ELECTRICAL NETWORK THEORY

By W. H. Ingram and C. M. Cramlet

Part I

By W. H. Ingram

Introduction. Part I essays an orderly arrangement of some elementary topological ideas in combination with some analytical ones of an essentially dynamical nature. A combination of the simplest examples from these two classes of ideas first appeared in the work of Kirchhoff and subsequent development of the theory of electrical networks has been, in fact if not in statement, an amalgamation of the two mathematical sciences. But the elementary theory, up until the present, has been incomplete in one particular, or at least has required clarification, namely in regard to the implication of the circuital flow of electricity from Kirchhoff's Second Law. This defect is removed in Theorem 4 where it is proved that the chord currents for any tree in the graph of a network have circuital properties.

The relation between the branch currents in a network and the circuital currents employed in their representation is commonly given by a rule which goes back to von Helmholtz and which assigns a circuital current to one mesh after another in the graph of a network until every branch is travelled by at least one such circuital current. This rule can be justified by an induction similar to that given by Brueckner for Euler's relation between the vertices, edges and faces of a polyhedron,[1] and in particular for the specialization of Euler's relation to a network of connected polygons in a plane, namely $V - B + N = 1$. Alternatively, there is the formula $N = B - V + 1$, one of the oldest in topology, for the cyclomatic number of Listing, whose derivation of the same may be made the basis of a more satisfactory, if not more general, proof of the von Helmholtz rule. As Quade has made clear,[2] it is possible to obtain a criterion as to the adequacy of a set of circuital currents for the representation of the branch currents by resort to the notion of the topological linear dependence of a set of circuits: if $C_1, C_2, \cdots, C_N$ is any set of linearly independent circuits in the graph of a network and if the inclusion of any other closed path whatever gives a properly augmented set which is not linearly independent, then the corresponding circuital currents $i_1, i_2, \cdots, i_N$ are adequate to the representation of the branch currents. A set of circuits $C_1, C_2, \cdots, C_N$ is said to be topologically linearly dependent if there exists integers $a_1, a_2, \cdots, a_N$ not all zero such that $a_1C_1 + a_2C_2 + \cdots + a_NC_N$ is a null-circuit, otherwise independent. The circuits must possess sense and it is necessary to define addition, subtraction and the null-circuit.[3] The circuital flow of the chord currents for any tree in the

[1] Brueckner, "Vielecke und Vielflaeche", Leipzig, 1900, p. 51.

[2] Quade, "Klassification der Schwingungsvorgaenge in gekoppelten Stromkreisen," Borna-Leipzig, 1933, Chapt.'s I, II.

[3] *Vide* Koenig, "Theorie der Endlichen und Unendlichen Graphen," Leipzig, 1936, Chapt. X.

graph of a network having been established in Theorem 4 of Part I, the von Helmholtz rule follows for any equivalent set of circuits $C_1, C_2, \cdots C_N$. This is proved by Cramlet in Part II.

The representation of the branch currents of the tree in terms of the chord currents which is implied by Kirchhoff's Second Law is thus the same as the representation obtained by von Helmholtz's rule on the assumption that the chord currents are circuital currents. The circuital currents may be said to be *generalized currents*, in the sense employed for velocities in dynamics, and the chord currents regarded as circuital currents accordingly afford a basis for all transformations in the linear manifold which they determine. This manifold contains at once the ranges and domains for the transformations and projections of currents distinguishable in time (e.g. the Fortescue resolutions) and the ranges and domains for the transformations considered by Cramlet in Part II where the currents are distinguishable topologically. It also contains, obviously, the domains if not the ranges of those little-understood transformations which have been interpreted as carrying one network into a topologically different one and an example of which has been given by Foster.

The chord charges become *generalized coordinates*, in the sense employed in dynamics, if we assume that they and the branch charges satisfy a set of linear homogeneous relations of constraint which, when differentiated with respect to the time, give the Kirchhoff equations, that is, if we assume that the vertices have no capacity.

A word must be said about the place of Kirchhoff's First[4] Law in the present theory. There are two objections to taking this "law" as a postulate for a theory of electrical networks. The first objection is that, in the form given by Kirchhoff, it is not general enough and the second is that it and its complete generalization can be derived from the fundamental equation (2) of Part I. For us, the latter equation has the status of a postulate.

1. **First Theorems.** Every electrical network has a topological representation called a *graph* consisting of two or more *vertices*, *branches* connecting them and separate or separable[5] *loops*. We shall assume that the number of vertices and branches is finite. A *sub-graph* is a graph contained in a graph.

The branches and loops represent the channels of electric currents in conductors which have the physical properties of resistance, inductance and electrical stiffness and which are the seats of the electromotive forces which originate the currents.

Corresponding to every vertex of a graph there is a Kirchhoff condition of constraint on the currents; thus the branch-currents are not independent. The number of independent constraints is always less than the number of vertices

[4] We follow the numbering finally chosen by Kirchhoff (Pogg. Annal., Vol. 72, 1847). Many text books follow the numerical designation of his laws given in his original communication (Pogg. Annal., Vol. 64, 1845).

[5] Examples of separable graphs are given by R. M. Foster, "Geometrical Circuits of Electrical Networks." Trans. Am. Inst. El. Engrs., 51, (1932) p. 309.

and is given by the rank of the matrix of coefficients in the Kirchhoff equations of constraint.

A fundamental theorem states that the rank of this functional matrix is numerically equal to the number of vertices V minus the number of separate parts P of the graph and is a consequence of the fact that every connected graph has a tree and every tree has one more vertex than branches.

DEFINITION. A *circuit* is a connected sub-graph the vertices of which can be arranged in cyclic order and the branches of which connect, and correspond with, one-to-one, each pair of vertices juxtaposed in the cycle. Specially, an isolated loop and a branch which is terminated at both ends by the same vertex are circuits.

DEFINITION. A *tree* is a connected sub-graph of a connected graph which consists of all the vertices of the graph and some or all of the branches but which contains no circuits.

THEOREM 1 (Von Staudt). *Every connected graph has a tree.*

PROOF. If the graph is not itself a tree, it contains at least one circuit and the same is true of the residual graph left after the removal of any branch in this circuit, and so on. Finally, a residual graph which connects all vertices of the graph proper must remain after the destruction in this way of all circuits in the graph.

DEFINITION. If a connected graph contains a proper sub-graph which is connected to its complement at but a single vertex, the graph is said to be *separable*. A vertex and a branch which leaves and rejoins the vertex are separable into a vertex and an *isolated loop*. An isolated loop may be a part of a graph.

DEFINITION. The branches complementary to a tree of a non-separable graph are called the *chords* to the tree.

THEOREM 2. *In any non-separable graph and for any branch of the graph a tree always exists which has the branch as a member and a tree always exists which has the branch as a chord.*

PROOF. By Theorem 1, the graph has at least one tree. If the branch is in the tree, the first part of the theorem is verified. If the branch is a chord, the sub-graph consisting of the tree and this chord has one circuit and this circuit is through the chord and at least one other branch of the sub-graph; the removal of this other branch makes the sub-graph into a tree and the first conclusion of the theorem is established.

If the branch is a chord, the second part of the theorem is verified. If the branch is a branch of the tree there must be a chord which, directly or indirectly, spans it, for otherwise the graph would be separable, contrary to hypothesis. The sub-graph consisting of the tree and the spanning chord reaches all vertices and contains one circuit. The removal of the branch converts this sub-graph into a tree with respect to which in the graph the branch in consideration is a chord and the second conclusion of the theorem is established.

THEOREM 3 (Von Staudt). *Every tree has one more vertex than branches.*

PROOF. Since a tree has no circuits, every path in the tree, no matter where commenced, must end in a cul-de-sac, i.e., every tree has terminal vertices (or

vertices terminating only one branch). If we remove one such terminal vertex and its connected graph branch, the residual part of the tree must have the same property, i.e. possess terminal vertices. Hence we may remove terminal vertices and branches, one at a time, until only one branch of the tree remains. But this branch must connect two vertices; hence the theorem.

COROLLARY. *The number of branches complementary to any tree in a connected graph* is $B - V + 1$.

THEOREM 4. *For any tree, the Kirchhoff equations of constraint imply circuital flow of the chord currents.*

PROOF. This theorem is a consequence of Kirchhoff's Second Law which says that the algebraic sum of all currents entering any vertex is zero.

Corresponding to any vertex, a constraint equation of this kind exists and can be solved for the current in one branch of the tree terminating there in terms of the currents in the chords and other possible branches of the tree terminating there. Each of said latter branches, if there are any, leads to other vertices corresponding to which an equation of constraint exists in each case and can be solved as in the first instance for the current in the branch from the first vertex in terms of the current in chords and other possible branches terminating in this next considered vertex, and so on. Finally, corresponding to every terminal vertex of the tree, an equation of constraint exists and can be solved for the current in the branch of the tree terminating there in terms of the current in the there connected chords. Hence the current in the branch first taken in the tree ultimately can be expressed in terms of chord currents exclusively, i.e. the current in any branch of the tree can be expressed in terms of the chord currents.

In particular, the current in any branch B_0 of a tree can be expressed in terms of the currents in the chords and tree-branches terminating in one of its vertices V_1. Let the current in B_0 be I_0 and let I_0 enter the vertex V_1. It may be assumed for the part of the tree which has V_1 as a vertex when B_0 is cut, and without loss of generality, that all branch currents have directions away from V_1 in the graph.

The Kirchhoff equations for V_1 accordingly have the form $I_0 = I_{1,1} + I_{1,2} + \cdots \pm i_{1,1} \pm i_{1,2} \pm \cdots$ and in general, for this part of the tree, there is the equation $I_{k-1} = I_{k,1} + I_{k,2} + \cdots \pm i_{k,1} \pm i_{k,2} \pm \cdots$ for the current entering the kth vertex expressed in terms of the branch currents entering neighboring vertices further along in the tree and in terms of the chord currents entering or leaving this vertex. For a terminal vertex V_1 of the tree, there is the equation $I_{1-1} = \pm i_{1,1} \pm i_{1,2} \pm \cdots$ with no branch current represented on the right-hand side.

Consider any chord connected with any two vertices in this part of the tree, say V_j and V_k, and let the corresponding chord current i' flow from V_j to V_k. Then in the equation for the jth vertex, i' will appear with a coefficient 1 and in the equation for the kth vertex with a -1. It follows that, after a series of substitutions in which the branch current entering any sink vertex is substituted into the Kirchhoff equation for the source vertex, and so on, the final expression of I_0 in terms of the chord currents can contain i_0 only as two terms with opposite signs. On the other hand, the current i'' for any chord which is connected to

only one vertex of this part of the tree will appear precisely once in the final expression and with a + sign if leaving the vertex and with a − sign if entering. Obviously this chord must span B_0 directly or span B_0 in series with other branches of the tree. The circuital flow of the chord currents is indicated.

Let i_0 be any chord current whatever and let its chord span the branches B_0, $B_1, \cdots, B_k$ in series and be connected with a vertex of B_0 and a vertex of B_k. By the preceding part of the proof, each of the corresponding branch currents $I_0, I_1, \cdots, I_k$ can be expressed as a corresponding linear combination with coefficients 1 or − 1, of chord currents and each expression must contain a term in i_0. Let $I_0', I_1', \cdots, I_k'$ be a set of currents equal in absolute value to $I_0, I_1, \cdots, I_k$ respectively and let their signs be chosen in agreement with the signs of the latter when the latter have directions in agreement with the direction of i_0 in the circuit formed by the chord carrying i_0 and the branches $B_0, B_1, \cdots, B_k$, and otherwise chosen in disagreement. Then there is a fundamental sequence of Kirchhoff equations $I_0' = I_1' + \cdots, I_1' = I_2' + \cdots, \cdots, I_j' = I_{j+1}' \cdots, I_k' = \cdots + i_0 + \cdots$, which are characterized by the presence of i_0 in the last expression and by its absence in all the others including the first (since the first equation of constraint is for that vertex of B_0 which is not a terminal of the chord). But a substitution of the last of these equation expressions into the preceding one, then a substitution of the result into the next to last, and so on, shows in turn that $I_{k-1}', I_{k-2}', \cdots, I_0'$ have each a component i_0. This proves the theorem.

COROLLARY. *No current can flow between a separable part of a network and its complement, nor in any branch which is not spanned directly or indirectly by some other branch, nor in a network whose graph is its own tree.*

COROLLARY. *The number of linearly independent currents in a network is at most the nullity*[6] *N of the network given by the equation* $N = B - V + P$, *where P is the number of separate or separable parts of the graph.*

2. **Dynamical developments.** For a connected graph, the nullity is $B - (V - 1)$ and hence the number of independent constraints is $V - 1$ at most. If any vertex of the connected graph is made a *ground point* then the remaining vertices correspond to $V - 1$ constraints. The *extended graph* formed by adjoining to the graph G the *Lagrangean connections*[7] between the ground point and the $V - 1$ other vertices obviously has $B + (V - 1)$ branches and nullity B if the Lagrangean connections are counted as branches. Since the Lagrangean connections radiate from the ground-point, we have

THEOREM 5. *The Lagrangean connections form a tree with respect to the extended graph.*

COROLLARY. *The number of independent constraints is* $V - 1$.

PROOF. There are $V - 1$ branches in any tree in the graph and $V - 1$ ver-

[6] The term *nullity* is due to Hassler Whitney, "Non-Separable and Planar Graphs," Trans. Am. Math. Soc., 34, (1932), p. 339; it is the nullity or degeneracy of the functional matrix.

[7] Ingram, Phil. Mag., 26, (1938), p. 270.

tices other than the ground point. Corresponding to each of these V − 1 vertices there is a branch of the tree leading to the ground point. There exists then a one-to-one correspondence between the Lagrangean connections and the branches of the tree. Considering the expressions for the currents in the branches of the radial tree formed by the Lagrangean connections themselves in the extended graph, it is seen that they are also in one-to-one correspondence with the branches of the tree.

DEFINITION. A *Blakesley step* for a directed quantity associated with some branch of a graph is the transference of the quantity into association with all other branches connected with one of the terminal vertices of the branch in such a way that the sense of direction of the directed quantity is preserved under the transfer with respect to all circuits in the graph through the branch.

DEFINITION. The *activity* of any force in a branch is the product of the force and the current in the branch.

THEOREM 6 (Blakesley). *The activity of an external e.m.f. under a Blakesley step is invariant.*

PROOF. As has been made clear in the proof of Theorem 4, the current in any branch of a tree can be expressed exclusively in terms of those chord currents which directly or indirectly span the branch and moreover expressed in agreement with the concept of the circuital flow of the chord currents. In any network, therefore, all the currents can be expressed algebraically and exclusively in terms of circuital currents which are either the chord currents assumed to flow in their circuits relative to the tree or are properly the loop currents of separable or isolated loops.

Any circuit in the network, in relation to any branch and terminal vertex of the network, either (a) excludes the branch and vertex, (b) includes both branch and vertex or (c) excludes the branch but not the vertex. In case (a) the activity of the e.m.f. on the circuital current is constantly zero. In case (b) the circuit must also include some other branch having the same terminal; the e.m.f. acts either in the first branch or in their other branch and in the same sense. In case (c) the circuit-current must enter the vertex the same number of times that it leaves. Before the step the activity of the e.m.f. on this circuit-current is zero because it does not act upon it; after the step the activity is effectively zero because in circuits through the vertex the e.m.f. transferred in association is transferred as many times in the negative sense as in the positive. This proves the theorem.

THEOREM 7. *A force acting in a branch of a tree of a network is equivalent to a set of equal forces acting in those chords which directly or indirectly span the branch.*

PROOF. We first establish a correspondence between any Blakesley step at a vertex and the process of equating one branch current to the other branch currents at the vertex in accordance with Kirchhoff's Second Law. If we disregard the direction of the current in any branch, a one-to-one correspondence obviously exists between any Blakesley step at the vertex and the process of equating one branch current there to the others in accordance with Kirchhoff's Second Law. Thus without regard to signs of terms, the equation, $I_1 = I_2 - I_3 + I_4$ corre-

sponds to a Blakesley step from some branch 1 into branches 2, 3 and 4, for example. By the proof of Theorem 4 the current in any branch of a tree can be expressed by a finite number of applications of Kirchhoff's Second Law in terms of the currents flowing in those chords which, directly or indirectly, span the branch in discussion. It follows that a finite number of Blakesley steps will bring the force first associated with this branch into final association with the chords of the theorem.

COROLLARY. *Any set of external e.m.f.'s is equivalent to a set of e.m.f.'s located in the chords to any tree of the graph.*

As we have seen, the chord currents have circuital properties. The idea, due to von Helmholtz,[8] of representing the motion of electricity in a network by a system of electrical currents flowing in circuits is thus almost, if not quite, implicit in Kirchhoff's 2nd Law and the concept of a graph and its topology. In the graph of any network and for any tree in the graph, we may assume a set of currents i_1, $i_2 \cdots$ flowing in closed polygonal circuits each of which consists of branches of the tree and one (corresponding) chord where its direction may be taken in agreement with the direction of the chord current; if I_1 is a chord current, then $I_1 = i_1$, etc. More generally, all the branch currents may be expressed in terms of the loop currents in the form

$$I = \Gamma i \tag{1}$$

where I is a columnar matrix whose elements, B in number, are the branch currents of the graph, Γ a $B \times (B - V + 1)$ rectangular matrix and i a columnar matrix whose $B - V + 1$ elements are the circuital currents. The elements of i are not necessarily linearly independent.

DEFINITION. The *first incidence matrix* of a graph and for any directed quantities associated one-to-one with the branches of the graph (e.g. the directions of the line-segments of the graph if possessing sense[9]) is a matrix Π in which the element in the i^{th} row and j^{th} column is $+1$ if the j^{th} directed quantity leaves the i^{th} vertex. -1 if the j^{th} directed quantity enters the i^{th} vertex and zero if the j^{th} quantity avoids the i^{th} vertex.

By definition, the incidence matrix Π is one such that Π I is a columnar matrix each of whose elements is the algebraic sum of the branch currents leaving the vertex corresponding to the position of the element in the column. By equation (1), $\Pi\Gamma$i is a columnar matrix each of whose elements is the algebraic sum of the circuital currents leaving the vertex in question. Therefore $\Pi\Gamma$ is an incidence matrix for the circuital currents. But the incidence of any circuital current at a vertex is both $+1$ and -1 if the vertex is in the circuit and 0 otherwise and we have, accordingly, the *Lemma* (Veblen[10]): *The* $V \times (B - V + 1)$ *matrix* $\Pi\Gamma$ *is one all of whose elements are zero.*

The analogy between the current flowing in a circuit and the velocity of a

[8] H. von Helmholz: "Poggendorff's Annalen," 83 (1851) p. 512.

[9] Poincaré, Proc. Lond. Math. Soc., 32 (1901), p. 280.

[10] *Cf* Veblen, *Analysis Situs* p. 65. It can be shown with the aid of certain results in Part II, that Γ is a matrix consisting of the linearly independent columns of Veblen's E_2.

material particle, established by Maxwell, also holds between the currents in the branches of a network and the velocities of a system of particles with kinematical connections. For every graph implies its extended graph with respect to which all its branches are chords and in which all its branch currents may be identified with currents flowing in circuits and in the Lagrangean connections of which the forces of constraint may be assumed to act to keep the algebraic sum of the last named circuital currents constantly zero in each said Lagrangean connections. For these circuital currents the forces of constraint have the nature of external forces: accordingly the equation of voltage for the k^{th} branch current has the form

$$E_k + V_{+k} - V_{-k} = \sum L_{kj} \frac{dI_j}{dt} + R_k I_k + S_k \int I_k \, dt \tag{2}$$

where V_k is the force of constraint associated with the *source* vertex for the current I_k, V_{-k} the force of constraint associated with the *sink* vertex for I_k and E_k is the external e.m.f. acting in the k^{th} branch.

The correspondence between the force of constraint associated with a vertex and the incidence of the branch currents at the vertex may be exhibited by means of the *constraint matrix* Λ where the j^{th} row indicates the force of constraint at the j^{th} vertex upon the branch currents locally identified with the circuital currents in the j^{th} Lagrangean connection and where at the same time the k^{th} column is the array of forces of constraint acting upon the k^{th} such circuital current. When the letter V_j is taken as the incidence of any branch current at the j^{th} vertex, then Λ may be thought of as a kind of incidence matrix: obviously $\Lambda = [V]\Pi$ where $[V]$ is a square diagonal matrix of elements $(V_1, V_2, \cdots, V_V)$.

LEMMA: *If* V *is the vector force of constraint on the network, then* Π_TV *is a columnar matrix whose* i^{th} *element is the algebraic sum of the forces of constraint on the current in the* i^{th} *branch of the network.*[11]

PROOF. Since $\Lambda_T = \Pi_T[V]$ is a matrix where the i^{th} row gives the incidence of the i^{th} current at the vertices of the Lagrangean connections (which are source and sink vertices for that current), the algebraic sum of the terms of that row gives the net force of constraint on that current. This algebraic sum is given by Π_TV where V is the result of compressing $[V]$ into a column, by a property of matrices.

It is readily verified that the equations (2), $k = 1, 2, \cdots, B$ are contained in the vector equation

$$E + \Pi_T V = [L] \frac{dI}{dt} + [R]I + [S] \int I \, dt \tag{3}$$

where Π_T is the transposed arrangement of Π, $[L]$, $[R]$, $[C]$ are square matrices, and where E and I are columnar matrices with the elements $E_1, E_2, \cdots, E_B$ and $I_1, I_2, \cdots, I_B$, respectively. As in the preceding lemma, V is a columnar matrix of elements $V_1, V_2 \cdots V_V$. Multiplication of (3) on the left by Γ_T gives a

[11] This theorem although known (*vide* Cauer, *Math. Annalen* 105 (1931), p. 90) seems not to have been stated nor proved before.

system of voltage equations free of the forces of constraint since $\Gamma_T\Pi_T = (\Pi\Gamma)_T$ is a matrix of zeros. The resulting equation

$$\Gamma_T E = \Gamma_T[L]\dot{Q} + \Gamma_T[R]Q + \Gamma_T[S]Q \tag{4}$$

where $\dot{Q} = I$, may be interpreted as a matric statement of a generalization of Kirchhoff's First Law to networks containing inductance and electrical stiffness as well as resistance. But in this equation there are more scalar unknowns than scalar equations in general and to solve for the unknown in (4) the equations of constraint $\Pi\dot{Q} = 0$ must be taken into account. More elegantly, we may make the substitution $\dot{Q} = \Gamma\dot{q}$ into (4) and take as our equations for the network the system

$$\begin{cases} e = [l]\dot{q} + [r]\dot{q} + [s]q \\ I = \Gamma\dot{q} \end{cases} \tag{5}$$

where $[l] = \Gamma_T[L]\Gamma$, $[r] = \Gamma_T[R]\Gamma$, $[s] = \Gamma_T[S]\Gamma$, $e = \Gamma_T E$ and where $i = \dot{q}$.

THEOREM 8. *When the external e.m.f.'s suddenly originate at time $t = 0$ in the chords to any tree of an electrically quiescent network, are all* L*-functions*[12] *of the time and are linearly independent, and when the impedance matrix has an inverse, then the chord currents are linearly independent.*

PROOF. It is known[13] that under these conditions both the e.m.f.'s and currents have Laplace transforms. Then equation (5) implies

$$e^*(\mu) = Z(\mu)\dot{q}^*(\mu) \tag{6}$$

a vector equation in which the columnar matrices $e^*(\mu)$, $q^*(\mu)$ have as their k^{th} components, respectively

$$e_k^*(\mu) = \int_0^\infty \epsilon^{-\mu t} e_k(t)\,dt, \qquad \dot{q}_k^*(\mu) = \int_0^\infty \epsilon^{-\mu t}\dot{q}_k(t)\,dt$$

the Laplace transforms of $e_k(t)$ and $q_k(t)$ respectively and where $Z(\mu)$ is the $N \times N$ impedance matrix $(l_{kj}\mu + r_{kj} + s_{kj}\mu^{-1})$.

If the components $e_1, e_2, \cdots, e_N$ of the columnar matrix $e(t)$ are linearly independent, then $a_1e_1(t) + a_2e_2(t) + \cdots + a_Ne_N(t) = \phi(t)$ where $\phi(t)$ is an L-function not identically zero for any set of constants $a_1, a_2, \cdots, a_N$ real *or* complex. Since every such function has a non-zero Laplace transform, the components of the vector $e^*(\mu)$ are linearly independent. If this is so, i.e. if there exists no line vector a with non-zero norm and components in the field of complex numbers such that the inner product $ae^*(\mu) = 0$ identically in μ, then there is no line vector bZ^{-1} of non-zero norm with components of the line-matrix b of elements b_k in the field of complex numbers such that the inner product $bZ^{-1}e^*(\mu) = 0$ identically in μ; that is there is no vector b of non-zero norm such that the inner product $bq^*(\mu)$ is zero identically. Hence the components of $\dot{q}^*(\mu)$ are linearly independent and it follows that the components of the vector $\dot{q}(t)$ are linearly independent, the proof being by contradiction.

[12] Doetsch, "Theorie u. Anwendung der Laplace-Transformationen," p. 13.
[13] Doetsch, p. 330.

COROLLARY. *The chord currents are necessarily independent when a set of e.m.f.'s active anywhere in the network is equivalent to a set of independent e.m.f.'s each an L-function of the time and active in all the chords, and are not necessarily independent when the equivalence does not hold.*

PROOF. The last part of the corollary follows from the fact that in a Wheatstone bridge of resistances with one external force all branch currents are linearly related and none is independent of any other.

3. On the Propagation of Blakesley Steps.

THEOREM 9. *In any non-separable graph, V steps are necessary to restore the relation of any directed quantity to the graph and nV sufficient, with n any natural number.*

PROOF. Let the directed quantity E_1 initially be in association with the branch B_1. By Theorem 2, there exists a tree in the graph with respect to which B_1 is a chord and there is a closed path through B_1 and completed in this tree. Blakesley steps made in order at the vertices of this path will transfer E_1 again into association with B_1, obviously. If the closed path has been across all the vertices of the graph the association is in two and nullifying senses in every chord and the directed quantity is in its initial relation to the graph. If fewer than V vertices are traversed, the quantity E_1 will be in association not only with B_1 but once with all branches of the graph having one terminal (and only one) a vertex of the path. One or more of these branches must belong to the tree in this case. Blakesley steps away from these vertices may be made in all such latter branches until, in all, one step has been made for each vertex in the tree and then the association is with no branch of the tree and is nullified in each every chord except B_1. The first step puts E_1 into association with a branch of the tree and a step for each branch of the tree is necessary to transfer the association out of the tree. By Theorem 3, the steps necessary to restore the relation of E_1 to the graph is thus V.

The proof of the second conclusion of the theorem is obvious when the closed path through the tree and B_1 crosses every vertex of the tree. When a proper part only of the tree is in the path and the path is gone over n times, each branch of the graph having a one-terminal connection with the path has E_1 brought into association with it n times and n Blakesley steps at each vertex are sufficient to produce n nullifications in each chord of the graph except B_1.

DEFINITION. An *external connection* of a graph is a branch having one and only one terminal which is a vertex of the graph.

THEOREM 10. *Any directed quantity associated with an external connection of a graph is transferred with sense reversed to all other external connections of the graph under V progressive Blakesley steps at most, V being the number of vertices of the graph, and after V such steps the association is only with the other external connections.*

PROOF. The vertices of any sequence of Blakesley steps form a path which cannot be re-entrant upon itself at any vertex for every branch connected with any vertex in the path except the branches of the path itself are in association

with the directed quantity. Hence the path contains no circuits. Since every vertex is connected to every other vertex in the graph and since any path from any vertex to any other is a possible path for a sequence of Blakesley steps it follows that there is no vertex not in the path when all possible steps have been taken and the path then is a tree. After V steps or less the quantity is transferred into association with all external connections if no step is taken back across the vertex of the preceding step. After the Vth forward step the quantity is in association only with the external connections for the association in each chord is once in each of two opposite senses which, from the definition of sense of a directed quantity, nullify each other.

COROLLARY. *The number of such nullifications is the nullity of the graph.*

COROLLARY. *The path of the steps is in a tree and the nullifications are in the chords.*

COROLLARY. *The relation of the directed quantity to the external connections after the V steps is the same as after one step when the graph is taken as coalesced to a point.*

COROLLARY. *The association is restored in all respects when any transferred directed quantity (in association with any one external connection) again is subjected to any V steps backward in the graph.*

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PART II

BY C. M. CRAMLET

Introduction. The graph of a network provides both a means for charting the flow of electricity in the network and at the same time a frame on which to attach the electrical constants. The matrix Π, as the arithmetical equivalent of the graph, can be made to serve in both these capacities with the aid of appropriate matrices of branch constants and the matrix Γ. But while the matrix Π and the matrices of branch constants are unique (to within the lettering of the vertices and branches) the matrix Γ is not and among the set of all possible trees and corresponding Γ's for any graph there is sometimes a particular tree having advantages over the others for the discernment of some particular property of the network. In other cases, the electrical state of the network is best described in terms of circuital currents whose circuits, although indirectly determined by every tree, are not directly determined by any. Nevertheless, in the general case, the adequacy of a basis of a set of circuital currents is more easily decided topologically than algebraically and the notion of a tree is a useful means to this decision.

The set of differential equations for the branches of the electrical network and the Kirchhoff equations of constraint determine the currents in all branches at all times. The branch-currents are also determined by the differential equations for the chord-circuital currents and the matrix Γ, as Part I has made clear. All trees in the graph are equivalent in this respect and in general there are equivalent sets of chord-circuital currents. Moreover, there are other sets of

circuital currents which are not chord-circuital currents but which determine the state of the network equally well (e.g., the von Helmholtz currents) and there arises the problem of the invariants of the class of the corresponding differential equations. In the course of a review of this problem, it became apparent that interesting results would probably come out of a development of a matric technique of circuit transformation whereby peculiarities of a network may be described in an appropriate choice of basis. Part II is concerned mainly with a technique for obtaining equivalent circuit bases and contains an application which may point to a broader utility of the technique developed.

1. **The first and second incidence matrices of a connected sagittal graph.** The element in the A^{th} row and k^{th} column of the first incidence matrix Π will be represented by Π_k^A, $A = 1, 2, \cdots, V$, $k = 1, 2, \cdots, B$. In later sections some significance may be attached to the position of an index but such interpretations will not apply to the matrices Π and Γ. The k^{th} column of Π contains two non-zero elements, namely a -1 in row P and a 1 in row Q indicating that B_k is directed $Q \rightarrow P$. The vertices V_P, V_Q are said to bound branch B_k and the boundaries of this branch are represented by the k^{th} column of Π. Any linear combination with integral coefficients of branches, called a chain, will have a boundary determined by the same linear combination of the corresponding columns of Π. For a closed chain, the linear combinations of these columns gives a column of zeros.[14] A linear combination of branches having this property and representing a closed chain will be represented by the letter C with a suitable subscript.

The columns of any matrix Π may be arranged so that the first $V - 1$ columns represent the branches of a tree and, since no linear combination of these branches can ever represent a circuit, the rank of Π must be at least $V - 1$. Since each of the then succeeding N columns represents a chord, the rank of Π must be exactly $V - 1$. Each of the latter columns determines a unique linear combination of branches representing a closed circuit C_α, $\alpha = 1, 2, \cdots, N$. The basis for the branches B_k, $k = 1, 2, \cdots, B$, may be replaced by the equivalent basis B_l, $l = 1, 2, \cdots, V - 1$ and C_α, $\alpha = 1, 2, \cdots, N$, the C's being evidently independent linear combinations of the B's for each contains a unique chord branch. Therefore there are precisely N independent linear combinations of these columns of Π. But to each linear combination of these columns corresponds a closed circuit C_α. Hence any other closed circuit is a linear combination of these. It follows that the C's constitute a *basis* for closed circuits. We have proved:

(1:1) *There is a unique set of basic circuits associated with a given tree in a sagittal graph.*

We shall take the expressions for the C's as linear combinations of the B's in the form

$$C_\alpha = \Gamma_\alpha^{*k} B_k . \tag{1:2}$$

[14] More details may be found in Seifert-Threlfall, "Lehrbuch der Topologie", Chapt. III, to be referred to as *S-T*.

In this equation and in all other equations dealing with the second incidence matrix Γ^*, the first incidence matrix Π, the matrix Γ, and in the general circuit theory, the following conventions will be made regarding the use of indices:

(1) A repeated letter will indicate a summation over the range of the letter used.

(2) Latin capitals take the range of vertices, namely 1, 2, $\cdots$, V.

(3) Small latin letters run over the range of branches, namely 1, 2, $\cdots$, B.

(4) Small greek letters take the range of circuits, namely 1, 2, $\cdots$, N; N = B $-$ V $+$ 1.

(5) An upper index indicates the row of a matrix and a lower index the column unless both indices are on the same level in which case the first and second are the row and column index, respectively.

The direction of C_α may be changed by multiplying the right-hand side of (1:2) by -1, that is, by multiplying the α^{th} column of Γ^* by -1. It may be convenient to do this in order to bring the direction of the circuit into agreement with the direction of the chord forming part of the circuit. Since C_α is a closed chain, its boundary vanishes (S$-$T), that is

$$BdC_\alpha = \Gamma_\alpha^{*k} Bd\mathrm{B}_k = 0. \tag{1:3}$$

The incidence matrix Π gives the boundary of B_k,

$$Bd\mathrm{B}_k = \Pi_k^A \mathrm{V}_A \tag{1:4}$$

hence

$$\Gamma_\alpha^{*k} \Pi_k^A \mathrm{V}_A = 0. \tag{1:4.1}$$

But the V's are a basis for vertices so

$$\Pi_k^A \Gamma_\alpha^{*k} = 0; \qquad \Pi\Gamma^* = 0. \tag{1:5}$$

The matrix Γ^* is of rank N. This is a consequence of equation (1:2) which expresses the N chains C_α as N linearly independent functions of the B_k. The elements of the first incidence matrix Π are 0, 1 and -1. When the elements of the second incidence matrix Γ^* are 0, 1 and -1, each circuit contains no branch more than once. We prefer to restrict our attention to such circuits. Consider the unsigned or neutral matrices $\bar{\Pi}$ and $\bar{\Gamma}$ obtained from Π and Γ^* above by replacing each element -1 by a 1. It is seen that the graph described by the new matrices is the original graph but with undirected branches, i.e. the neutral graph. The reverse operation of passing from the neutral incidence matrix $\bar{\Gamma}$ to Γ^* may be done by first assigning sense to all branches in an arbitrary manner, then assigning sense to each circuit C_α. In going around the circuit in the positive sense, the element Γ_α^{*k} will be 1 or -1 according as the k^{th} branch is directed in agreement with C_α or oppositely. The modular ranks of the neutral incidence matrices $\bar{\Pi}$ and $\bar{\Gamma}$ are V $-$ 1 and N, respectively, as may be seen by a proof like that for the corresponding theorem for Π and Γ^*.

From (1:2) it appears that the sum of two circuits is given by an equation

$$\bar{C}_\alpha + \bar{C}_\beta = (\bar{\Gamma}_\alpha^k + \bar{\Gamma}_\beta^k)\mathrm{B}_k$$

where the over-lining indicates the neutral graph. A new circuit matrix $\bar{\Gamma}'$ may therefore be obtained by replacing the α^{th} column by the sum of the α^{th} and the β^{th} columns and in general, when operations are restricted modulo 2, to elementary operations on columns. If the two added circuits have branches in common, the sum will be a circuit having the boundary of both circuits not including the common branches. If there are no common branches, the resulting circuit will be composed of two closed loops. This latter type of circuit will be called *disjoint*, the former consisting of a single closed loop will be called *simple*.

Let the branches of the graph be so re-lettered that the first N are chords and B_α is the chord of the circuit C_α. Since each chord is on one and only one circuit, the first N rows of $\bar{\Gamma}$ form a matrix with 1's on the diagonal and 0's elsewhere. The remaining rows will then describe the tree. The set of undirected or neutral circuits determined by a tree will be called a *chord frame*. When, corresponding to an appropriate numbering of the branches, the first N rows of $\bar{\Gamma}$ is a unit matrix, the chord frame will be said to be a *canonical chord frame*. Trees may be selected in a graph in various ways and hence it follows that there is a class of chord frames associated with any graph. It is characteristic of a chord frame that each chord is associated one-to-one with a simple circuit.

It may be shown, as in the proof of theorem (1:1), that any N independent undirected circuits constitute a basis for all undirected circuits in a graph. We may select as such a basis a canonical chord frame. Then all circuits no one of which includes any branch more than once are obtainable by independent linear combinations of these *modulo* 2, i.e.,

$$\bar{C}'_\alpha = \bar{a}^\beta_\alpha \bar{C}_\beta, \quad det\ \bar{a} = 1, \ modulo\ 2. \tag{1:6}$$

The new basis, not in general a chord frame, will be called a *coordinate frame*.

The number M of non-singular $N \times N$ matrices $\bar{a}$ is

$$M = (2^N - 1)(2^N - 2)(2^N - 2^2) \cdots (2^N - 2^{N-1}). \tag{1:7}$$

These matrices form a group $\mathfrak{G}_M$ of order M under the operation of multiplication modulo 2. The number of frames, not distinguishing the N! ways of numbering them, is

$$\mu = M/N! \tag{1:7.1}$$

The number of chord frames is less, in general, than the number of coordinate frames. For example, if the graph is the tetrahedron ABCD, a coordinate frame is given by the circuits ABCDA, ABDCA and ADCA. Every branch of this third loop is on one or more of the first and second loops so, although these circuits constitute a coordinate frame, they do not constitute a chord frame.

The following theorem states a property of chord frames:

(1:8) *Two circuits of a chord frame have at most one connected chain in common.*

The theorem may be proved by making the contrary assumption, namely that the connected chains K and K′ are each common to the two circuits and have no vertex in common. Let V and V′ be vertices at ends of these respective chains. Since these are on the first circuit C_1 they are connected through the tree by a

chain K_1 on C_1. Not all branches of K_1 can be on the second circuit C_2 by hypothesis. Similarly, these vertices are connected by a chain K_2 on C_2 and in the tree but not wholly on C_1. Therefore $K_1 \neq K_2$ and $K_1 + K_2$ forms a closed chain or circuit. But a tree contains no circuits and the theorem follows.

By theorem (1:1), the directed circuit C'_α corresponding to $\bar{C}'_\alpha$ is linearly related to the circuit C_β corresponding to the undirected $\bar{C}_\beta$, i.e.

$$C'_\alpha = a^\beta_\alpha C_\beta. \tag{1:9}$$

Equations (1:6) and (1:9) are topological equations but they correspond to the numerical equations

$$\bar{\Gamma}' = \bar{\Gamma}\bar{a}, \textit{ modulo } 2, \tag{1:10}$$

and

$$\Gamma^{*\prime} = \Gamma^* a, \tag{1.11}$$

respectively. Since Γ^* and $\Gamma^{*\prime}$ correspond to $\bar{\Gamma}$ and $\bar{\Gamma}'$, that is, by the direction assigning process

$$\bar{\Gamma} \rightarrow \Gamma^*, \qquad \bar{\Gamma}' \rightarrow \Gamma^{*\prime}, \tag{1:12}$$

and the matrices Γ^* and $\Gamma^{*\prime}$ have elements 1, 0 and -1; the correspondences (1:12) imply

$$\bar{\Gamma} = \Gamma^*, \bar{\Gamma}' = \Gamma^{*\prime}, \textit{ modulo } 2. \tag{1:13}$$

Here the $\bar{C}$'s constitute a canonical chord frame so that the first N rows of $\bar{\Gamma}$ form a unit matrix γ. With a similar notation for the other Γ's, (1:10) and (1:11) give

$$\bar{\gamma}' = \bar{a}, \textit{ modulo } 2, \qquad \gamma' = a, \tag{1:14}$$

and from (1:13), $\bar{\gamma}' = \gamma'$, so that

$$a = \bar{a}, \textit{ modulo } 2. \tag{1:15}$$

From (1:10) and (1:11), $\bar{\gamma} = \bar{\gamma}'(\bar{a})^{-1}$, *modulo* 2, $\gamma = \gamma' a^{-1}$. The left members of these equations are equal to each other and since $(\bar{a})^{-1}$ is a matrix with elements 1, 0, it follows that $\bar{\gamma}'(\bar{a})^{-1} = \gamma'(\bar{a})^{-1} = \gamma'(a)^{-1}$, *modulo* 2, so that

$$\gamma'[(\bar{a})^{-1} + (a)^{-1}] = 0, \textit{ modulo } 2. \tag{1:16}$$

By (1:6), (1:10) and the second equation of (1:13), $det\ \gamma' = 1$, *modulo* 2 and γ' has an inverse; hence equation (1:16) implies

$$(a)^{-1} = (\bar{a})^{-1}, \textit{ modulo } 2. \tag{1:17}$$

Equations (1:15) and (1:17) show that a and its inverse are matrices of integers. From this and the fact that $det\ (a^{-1}a) = 1$, it follows that

$$det\ a = \pm 1. \tag{1:18}$$

A directed circuit basis will be called a *coordinate system*. Each α relates a canonical coordinate system to a coordinate system. Let α_1 take C into C_1, α_2 take C into C_2 and α_{12} take C_1 into C_2, then

$$\alpha_{12} = \alpha_2\alpha_1^{-1} \tag{1:19}$$

is also a matrix of integers with determinant equal to ± 1. From the second equation of (1:14) it appears that the determinant of any α may be made equal to 1 by a change in sign of a column of Γ^*, that is, by a reversal of the sense of a circuit, if necessary. Then, by (1:19), all α's will have a determinant equal to 1. The matrices $\bar{\alpha}$ define a group of neutral bases by means of equations (1:6).

The principal results of these calculations are summarized in the theorem

(1:20) *Any two coordinate systems are related by equations like* (1:9) *with the matrix* α *composed of integers* 1, -1, 0 *and such that its determinant is equal to* ± 1.

It is apparent that the matrices α described in this theorem form a set that is a multiple isomorphism of the group $\mathfrak{G}_M$. The correspondence is given by (1:15). A coordinate system offers a suitable reference system for a description of the state of the network. Current, e.m.f. and the circuit constants are defined in this directed basis. Transition from one coordinate system to another can be made through coordinate frames. All coordinate frames are given by $\mathfrak{G}_M$. The associated coordinate systems are determined by the sign-assigning process. A deviation from what is usual is to be noted in the fact that the set of frames may contain disjoint circuits and, if the network contains at least two circuits having no common branch, will in fact contain such circuits.

In the correspondence (1:13), the circuits may be so directed that the first N rows of Γ^* also form a unit matrix. Such a matrix will be said to determine a *canonical coordinate system*.

2. Generalization of the circuital flow theorem. The Kirchhoff equations of constraint on the currents I^k of a sagittal graph are

$$\Pi_k^{\Delta} I^k = 0 \tag{2:1}$$

Since the matrix Π is of rank $V - 1$, this number of the branch currents may be solved for the remaining N. Let this be done in any manner and re-letter the currents, if necessary, so that the parametric branch currents are $I^\alpha = i^\alpha$. Then all branch currents I^k may be expressed as functions of the i^α by the equations

$$I^k = \Gamma_\alpha^k i^\alpha. \tag{2:2}$$

If the branches containing the i^α are cut, that is, if every $i^\alpha = 0$, no current can flow in the network since these equations show that every $I^k = 0$. It would seem then that the i^α are the chord currents. This will be shown to be in fact the case. The equations (2:2) are solutions of (2:1) for arbitrary values of i^α, that is, from $\Pi\Gamma i = 0$ it follows that $\Pi\Gamma = 0$. From (1:3.4), $\Pi\Gamma^* = 0$. These two matric equations lead to an identifying of Γ and Γ^*.

Consider the system of V equations in B unknowns

$$\Pi^{\Delta}_{k} x^{k} = 0. \tag{2:3}$$

These will be written in the form

$$\Pi^{\Delta}_{\alpha} x^{\alpha} + \Pi^{\Delta}_{\alpha'} x^{\alpha'} = 0, \tag{2:4}$$

where α takes the usual range $1, 2, \cdots, N$, but α' takes the range $N+1, N+2, \cdots, B$. The tree matrix $\Pi^{\Delta}_{\alpha'}$ has the rank $V-1$. If the set of x's for $\alpha = 1, 2, \cdots, N$ are arbitrarily assigned, there will be a unique solution if the V equations are satisfied. This condition assures consistency for there is one more equation than the rank of the set. Now put $x^{\alpha} = \Gamma^{*\alpha}_{\beta}$, $\alpha = 1, 2, \cdots, N$, β fixed, where Γ^{*} determines a canonical coordinate system. But this β^{th} column of the matrix Γ^{*} is a solution of (2:3) as is expressed by (1:5). This proves that the matrix Γ^{*} is uniquely determined by the equation $\Pi\Gamma^{*} = 0$ when the first N rows of Γ^{*} are taken in canonical form. But the first N rows of Γ and Γ^{*} are precisely alike; hence $\Pi\Gamma = 0$ implies

$$\Gamma = \Gamma^{*} \tag{2:5}$$

for a canonical coordinate system.

This demonstrates that the α^{th} column of Γ gives all branches through which we may regard the current i^{α} as flowing and that these branches form the closed circuit of Theorem 4, Part I. That is, the circuital currents are measurable in the chords.

The proof will now be extended to any other circuit choice under the group $\mathfrak{G}_{M}$. We may now consider Γ of (2:2) as the canonical incidence matrix and the i's as the loop currents of the corresponding circuits C. Any other coordinate system C′ may be given by a suitable matrix $\mathfrak{a}$ in equation (1:9). Accordingly

$$\Gamma'^{k}_{\alpha} = \Gamma^{k}_{\beta} \mathfrak{a}^{\beta}_{\alpha} \tag{2:6}$$

and these equations may be solved for the Γ's. Specifically,

$$\Gamma^{k}_{\alpha} = \Gamma'^{k}_{\beta} \mathfrak{A}^{\beta}_{\alpha} \qquad \mathfrak{a}\mathfrak{A} = 1. \tag{2:7}$$

Then (2:2) yields the equations

$$I^{k} = \Gamma'^{k}_{\alpha} i'^{\alpha} \tag{2:8}$$

$$i'^{\alpha} = \mathfrak{A}^{\alpha}_{\beta} i^{\beta} \tag{2:9}$$

where Γ' is for any coordinate system and the i'^{α} are the appropriate circuital currents. This completes the generalization of the circuital flow theorem. The e.m.f's and circuit constants will now be defined for the general coordinate system.

When the equations (5), Part I, are written explicitly they take the form

$$\begin{aligned} e_{\alpha} &= l_{\alpha\beta} \ddot{q}^{\beta} + r_{\alpha\beta} \dot{q}^{\beta} + s_{\alpha\beta} q^{\beta}, \\ e_{\alpha} &= \Gamma^{k}_{\alpha} E_{k}, \qquad i^{\alpha} = \dot{q}^{\alpha}, \end{aligned} \tag{2:10}$$

and Γ has been shown to be the second incidence matrix if the coordinates are a chord system.

We shall call equations (2:6) a *transformation of coordinates*. Then the circuital currents have transformed under the induced transformation (2:9), that is, the q's, $\dot{q}$'s and $\ddot{q}$'s transform like the i's in (2:9). For the new coordinate system,

$$e'_\alpha = l'_{\alpha\beta}\ddot{q}'^{\beta} + r'_{\alpha\beta}\dot{q}'^{\beta} + s'_{\alpha\beta}q'^{\beta}, \tag{2:11}$$

$$l'_{\alpha\beta} = l_{\rho\sigma}a^\rho_\alpha a^\sigma_\beta, \qquad r'_{\alpha\beta} = r_{\rho\sigma}a^\rho_\alpha a^\sigma_\beta, \qquad s'_{\alpha\beta} = s_{\rho\sigma}a^\rho_\alpha a^\sigma_\beta, \tag{2:12}$$

$$e'_\alpha = a^\beta_\alpha e_\beta. \tag{2:13}$$

It is necessary that the e's transform by equation (2:13) to satisfy the physical requirement that the activity

$$A = e_\alpha i^\alpha \tag{2:14}$$

be invariant. As a consequence of (2:9) and (2:12), the kinetic energy T, potential energy V and dissipative function F are all invariant:

$$T = \tfrac{1}{2}l_{\alpha\beta}\dot{i}^\alpha\dot{i}^\beta$$

$$V = \tfrac{1}{2}s_{\alpha\beta}q^\alpha q^\beta \tag{2:15}$$

$$F = \tfrac{1}{2}r_{\alpha\beta}i^\alpha i^\beta,^{15}.$$

A consequence of the second equations of (2:10), (2:13) and (2:7) is the equation

$$e'_\alpha = \Gamma'^k_\alpha E_k, \tag{2:16}$$

that is, e'_α is the sum of all applied e.m.f.'s in the α^{th} circuit for any coordinate system, even for a disjoint circuit. If the new coordinate system is a chord system, the new currents may be measured in the chords, otherwise it is expedient to use (2:9) as a definition of the new currents. Similarly, for a chord system, the e.m.f.'s can be taken to be in the chords, as in Part I. In a general coordinate system, they may be determined intrinsically by (2:16) or inferred from the chord system by (2:13).

3. **Dependent equations on independent circuits.** The general graph has been seen to admit a selection of N independent topological undirected circuits in μ ways, μ given by (1:7.1). By the process of assigning sense, a coordinate system may be associated with each frame. When the electrical variables and parameters are associated with a coordinate system, the equations (2:10) have significance. The question of the dependence of these differential equations in a proper N dimensional coordinate system is the subject of this section.

In the most general case, the differential equations (2:10) determine (geometrically speaking) a curve through any point of this N dimensional space and in any direction. However, the system of differential equations may be

[15]Maxwell, Electricity and Magnetism, 3rd Ed., Vol. II, p. 223; Rayleigh, Phil. Mag., 21, (1886), p. 369.

dependent and the curve lie in a space of lower dimension. If, for example, a single linear relation exists among the differential equations, the following relations will exist simultaneously:

$$(3:1) \qquad k^{\alpha} l_{\alpha\beta} = 0, \quad k^{\alpha} r_{\alpha\beta} = 0, \quad k^{\alpha} s_{\alpha\beta} = 0, \quad k^{\alpha} e_{\alpha} = 0.$$

Let us assume that the network is of such type as to admit a set of chord circuits (coordinate system) such that no circuit has more than one branch in common with any other circuit. Then

$$(3:2) \qquad 2r_{\alpha\alpha} \geq \Sigma_{\beta} |\, r_{\alpha\beta} \,|, \qquad \alpha = 1, 2, \cdots, N.$$

If, moreover, equations like (3:2) restrict l and s as well as r, the network will be said to be an *R-type network*.[16] That there is indeed a limitation here may be seen from the following simple illustration of a network composed of two vertices and four branches joining them. The two matrices

$$\Gamma_T = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix}, \qquad \Gamma'_T = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

determine coordinate frames for this network. The first satisfies (3:2) but is not a chord system. The second is a chord system but does not satisfy (3:2) when the resistance in the fourth branch (corresponding to the fourth column) is larger than the resistance of any other branch.

For an R-type matrix we have the theorem:

(3:3) *If the differential equations of an R-type coordinate system are linearly dependent, then the linear relations have coefficients* 1, −1 *and* 0 *in the defined coordinate system and there are as many independent simple circuits with all self and mutual* l, r *and* s *which are zero as there are independent solutions of the first three sets of* (3:1).

This theorem says that if the number of independent equations, differential and finite, and including the V − 1 independent Kirchhoff equations of constraint, is B − τ, then there are τ topologically independent circuits devoid of all l, r and s, self or mutual. In clarification, attention may be restricted to circuits containing resistance only. From the hypotheses of (3:3), we have in this case

$$(3:4) \qquad 2r_{\alpha\alpha} \geq \Sigma_{\beta} |\, r_{\alpha\beta} \,|, \qquad r_{\alpha\beta} x^{\beta} = 0.$$

If $|\, x^{\alpha} \,|$ is a maximum among the absolute values of the x's, then

$$(3:5) \qquad 2r_{\alpha\alpha} |\, x^{\alpha} \,| > |\, r_{\alpha\beta} x^{\beta} \,|, \ \alpha \text{ not summed, } \beta \text{ summed.}$$

That is, if the term $r_{\alpha\alpha} x^{\alpha}$ dominates the sum $r_{\alpha\beta} x^{\beta}$, then $r_{\alpha\beta} x^{\beta} \neq 0$, contradicting the hypothesis. Hence there cannot be a maximum $|\, x^{\alpha} \,|$. With a suitable notation, the alternatives are

$$(3:6.1) \qquad |\, x^1 \,| = |\, x^2 \,| = \cdots = |\, x^{\kappa} \,| > |\, x^{\nu} \,|, \quad \nu = \kappa + 1, \kappa + 2, \cdots, N, \quad \kappa = 2, 3, \cdots, N - 1,$$

$$(3:6.2) \qquad |\, x^1 \,| = |\, x^2 \,| = \cdots = |\, x^{\alpha} \,|, \qquad \alpha = 2, 3, \cdots, N.$$

[16] Burington, "R-Matrices and Equivalent Networks," Journ. Math. and Physics, 16 (1937), p. 85.

It is assumed in the equations of (3:4) that not all x's vanish. The alternative (3:6.2) leads to the solution in which every $|x^\beta| = 1$. In considering the alternative (3:6.1), let the index α be assumed among the indices $1, 2, \cdots, \kappa$. Then (3:5) will hold and lead to an inconsistency unless $r_{\alpha\beta} = 0$, $\beta = \kappa + 1, \kappa + 2, \cdots, N$. The matrix r is now reduced to two square matrices r′, r″ of orders κ and $N - \kappa$, respectively, and the form to

$$r_{\alpha\beta}x^\alpha x^\beta = r'_{\alpha\beta}x'^\alpha x'^\beta + r''_{\alpha\beta}x''^\alpha x''^\beta. \tag{3:7}$$

The matrices r′ and r″ are also semi-positive definite and solutions of (3:4) can only be found by finding solutions of the independent sets $r'_{\alpha\beta}x'^\beta = 0$, $r''_{\alpha\beta}x''^\beta = 0$. The arbitrary factors associated with the respective solutions of these sets of homogeneous equations are independent. Treating these sets individually in a like manner, new sets of semi-definite forms are obtained and the solution of (3:4) resolves to the solution of the independent sets

$$r^{(k)}_{\alpha\beta}x^{(k)\beta} = 0, \quad k = 1, 2, \cdots, K, \quad \alpha, \beta = 1, 2, \cdots, N^{(k)}, \quad \Sigma N^{(k)} = N. \tag{3:8}$$

That is, the matrix consists of K blocks along the diagonal. For a non-singular block (regarded as a matrix of order $N^{(k)}$) all of the associated members $x^{(k)\beta}$ of a solution will be equal to zero. A singular block will be or rank $N^{(k)} - 1$ with all members of the solution having the same absolute value. It appears. then, that the number of fundamental solutions equals the number of singular blocks and that each of these fundamental solutions may be selected as the solution of a block with all x's which are not zero conveniently taken with absolute values equal to 1 and all other x's not associated with this block taken equal to zero.

Consider any singular block of r. An inequality like (3:4) will hold and it follows that

$$2r_{\alpha\alpha}|x^\alpha| \geq |r_{\alpha\beta}||x^\beta|, \qquad |x^\alpha| = 1, \tag{3:9}$$

β summed over the columns of the block, and $2r_{\alpha\alpha}|x^\alpha| \geq |r_{\alpha\beta}x^\beta|$, or $r_{\alpha\beta}x^\beta$ has the same sign as x^α. For this to be consistent with (3:8), the equality sign must be selected here and in (3:9). The inequalities (3:9) then take the form

$$r_{\alpha\alpha}|x^\alpha| = |r_{\alpha\beta}||x^\beta|, \qquad \beta \neq \alpha, \tag{3:10}$$

and writing (3:8) in the form $-r_{\alpha\alpha}x^\alpha = r_{\alpha\beta}x^\beta$, where $\beta \neq \alpha$, it appears by comparison that every term on the right of the last equation must have the same sign as the left member and that if $r_{\alpha\beta} > 0$ then x^α and x^β must differ in sign. If $r_{\alpha\beta} < 0$, they have the same sign.

We now have a solution of the equations (3:4) in which the x's which are associated with this singular block have values 1 and -1 and the remaining x's each have the value zero. The blocks can be used in the construction of a transformation matrix. For convenience, this singular block will be considered to be in the first position of the matrix r. The solution x will be used to determine the first column of a non-singular transformation matrix a for the matrix r of (2:12) and the remaining columns will consist of 1's on the diagonal and zeros elsewhere. The transformation coefficients for the circuits (2:6) and the

circuit constants (2:12) are the same, both transforming cogrediently. On the assumption that the circuit basis is a chord system and that $r_{\alpha\beta} \neq 0$, circuits C_α and C_β are opposed or in agreement in sense in their common connected chain (Theorem 1:8) according as $r_{\alpha\beta} < 0$ or $r_{\alpha\beta} > 0$, but x^α and x^β, and hence a_1^α and a_1^β, have like or unlike signs in the corresponding cases. The second equation of (3:4) now becomes $r_{\alpha\beta}a_1^\beta = 0$ an equation which, from (2:12), implies that $r'_{1\alpha} = 0$, $\alpha = 1, 2, \cdots, N$. This corresponds to a transformation $C'_1 = a_1^\alpha C_\alpha$, belonging to the set (1:9): for every branch that appears in this sum, as in (1:10), appears either once or twice (from the definition of an R-type network) and if appearing twice will appear with opposite signs and cancel. The effect is the same as if the addition were modulo 2. Let V be a vertex terminating a branch B that is common to circuits C_1 and C_2. Let B_1 of C_1 and B_2 of C_2 be two other branches incident at V. If the direction of $a_1'C_1$ is such that V is a sink vertex for B_1, then $a_1^2C_2$ will be directed so that V is a source vertex for B_2 and both branches will be components of the circuit $a_1'C_1 + a_1^2C_2$ with directions in agreement with the sense of the circuit.

For an R-type network, the two-parameter matrix $m = 1 + \lambda r + \mu s$ may be carried through the proof in place of r. Equations such as (3:4) will give three sets of inequalities and the x's will be simultaneous solutions of three sets of equations. The final step will lead to the exhibition of a circuit C'_1 such that $m'_{1\beta} = 0$, that is, such that $l'_{1\beta} = r'_{1\beta} = s'_{1\beta} = 0$. Let ρ be the rank of m. There are then $\tau = N - \rho$ singular blocks of m and as many independent circuits with vanishing circuital electrical constants. Some of these circuits may be disjoint. Let the number of independent simple circuits forming a set equivalent to this set of τ circuits be σ. Adjoin to these $N - \sigma$ other circuits to form a basis. But the rank of m is invariant under a change of basis so that $\rho = N - \sigma$ and therefore $\sigma = \tau$. This concludes the proof of theorem (3:3).

A distinction will be made between an R-type and a *resistance type* network. The latter will be defined to be a network in which all mutual elements of any two circuits reside in branches common to these circuits. A dependence theorem for this type will now be established. The resistance matrix (Part I, eqn. 5) is described in terms of the directed or signed matrix Γ by the equations

$$r_{\alpha\beta} = R_{st}\Gamma^s_\alpha \Gamma^t_\beta . \tag{3:11}$$

A coordinate frame is the neutral circuit basis of a coordinate system. In a coordinate frame, these equations take the form

$$\bar{r}_{\alpha\beta} = R_{st}\bar{\Gamma}^s_\alpha \bar{\Gamma}^t_\beta . \tag{3:12}$$

As a consequence of the diagonal character of the matrix R and since all branch elements are distinct indeterminates, these equations give

$$r = \bar{r} \quad \text{modulo } 2 \tag{3:13}$$

an equation taken to mean that all diagonal elements of r and $\bar{r}$ are exactly alike in the self-resistances and that the off-diagonal terms of $\bar{r}$ are obtained from r by replacing each minus sign associated with a branch element by a plus sign.

Under a transformation of coordinate frames (1:6), the matrix $\bar{\mathfrak{r}}$ is transformed analogously to (2:12) by the equations

$$\mathfrak{r}'_{\alpha\beta} = \mathfrak{r}_{\rho\sigma}\bar{a}^{\rho}_{\alpha}\bar{a}^{\sigma}_{\beta}. \tag{3:14}$$

If $\bar{\mathfrak{r}}$ is singular modulo 2, there exists a set $\bar{a}^{\rho}_{i}$ equal to 0 or 1 such that $\mathfrak{r}_{\rho\sigma}\bar{a}^{\rho}_{i} = 0$, modulo 2. To this set may be adjoined others to construct a non-singular transformation matrix $\bar{a}$.

In the new coordinate frame, $\mathfrak{r}'_{i\beta} = 0$ modulo 2 and in the associate coordinate system $r'_{i\beta} = 0$, $\beta = 1, 2, \cdots, N$, that is

(3:15) *The necessary and sufficient condition that a resistance-type network have a circuit containing only zero circuit constants is that the matrix* $\bar{\mathfrak{m}} = \bar{\mathfrak{l}} + \lambda\bar{\mathfrak{r}} + \mu\bar{s}$, λ, μ *arbitrary, be singular modulo* 2.

It is now possible to state a theorem for resistance-type circuits that is similar to theorem (3:3):

(3:16) *The rank of a system of differential equations of a network is equal to the total number of independent circuits diminished by the number of independent circuits in which all circuital electrical constants vanish.*

This rank can be calculated by counting any branch containing vanishing electrical constants as a vertex in the construction of the incidence matrix Π. But it would appear that no branch containing an e.m.f. can be treated in this manner.

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COEFFICIENTS FOR INTERPOLATION WITHIN A SQUARE GRID IN THE COMPLEX PLANE

By Arnold N. Lowan and Herbert E. Salzer[1]

If it is known that to the desired degree of accuracy the Taylor expansion of an analytic function $f(z)$ in the neighborhood of a point z_0 may be truncated after $n+1$ terms, this means that in the neighborhood of z_0 the function may be approximated by an n-degree polynomial in $\Delta z = z - z_0$ and therefore by an n-degree polynomial in z. An approximating polynomial may also be obtained from the general Lagrange-Hermite interpolation formula:

$$f(z) = \sum_{\nu=m}^{m+n} \frac{P_\nu(z)}{P_\nu(z_\nu)} f(z_\nu) + R_n \quad \text{where}$$

$$P_\nu(z) = \frac{1}{z - z_\nu} \prod_{i=m}^{m+n} (z - z_i) \quad \text{and}$$

$$R_n = \frac{(z - z_m)(z - z_{m+1}) \cdots (z - z_{n+m})}{2\pi i} \int_C \frac{f(t)}{(t - z_m) \cdots (t - z_{m+n})(t - z)}\, dt$$

where the points z_ν are chosen in the region of validity of the truncated Taylor expansion and the contour C surrounds the points z_ν and z.

Making use of the integral representation of the $(n+1)$th divided difference and of a theorem due to Paul Montel[2], the remainder R_n may be written in the form

$$(z - z_m)(z - z_{m+1}) \cdots (z - z_{m+n}) f^{(n+1)}(\alpha)/(n+1)!$$

where $f^{(n+1)}(\alpha)$ is a point within the domain of convexity corresponding to the domain covered by $f^{(n+1)}(z)$ when z covers a convex region containing the points z_ν and z. In practice it usually suffices to take $f^{(n+1)}(z_\nu)$ for any z_ν, in place of $f^{(n+1)}(\alpha)$.

In the case where the z_ν's and z are real and the z_ν's are equidistant, the expressions $P_\nu(z)/P_\nu(z_\nu)$ reduce to the well-known interpolation polynomials $A_\nu^{(n)}(p)$ where $p = (x - x_0)/h$ and h is the tabular interval of the function.[3] More generally when the z_ν are equidistant points along any straight line in the z-plane the expressions $P_\nu(z)/P_\nu(z_\nu)$ reduce to polynomials identical with the $A_\nu^{(n)}(p)$ except that p denotes the ratio $(z - z_0)/H$ where H is the complex tabular interval.

When $f(z)$ is known over a square grid in the z-plane (for equidistant values of x and y) and when it may be approximated by a quadratic or cubic polynomial, it is convenient to choose the points z_ν at the corners of a square whose sides are parallel to the real and imaginary axes and of length h. The complex arguments

[1] Mathematical Tables Project, National Bureau of Standards, New York City.

[2] "Sur quelques propriétés des différences divisées." Journal de Mathématique Pures et Appliquées, 9 Serie, 16, 1937, pp. 219–231.

[3] These polynomials have been extensively tabulated in the "Table of Lagrangian Interpolation Coefficients," Columbia Press, 1944.

associated with these corners will be denoted by z_{-1}, z_0, z_1 and z_2 where z_{-1} corresponds to the upper left corner and the other arguments are associated with the remaining three corners taken in counter-clockwise order. If $z = z_0 + \Delta z$ is some point within or upon the side of the square, $\Delta z = Ph = (p + iq)h$ where $0 \leq p \leq 1$ and $0 \leq q \leq 1$. When $f(z)$ may be approximated by a quadratic polynomial in z, we shall assume that $f(z)$ is given for the points z_{-1}, z_0 and z_1; when $f(z)$ may be approximated by a cubic polynomial we shall assume that $f(z)$ is given at the arguments z_{-1}, z_0, z_1 and z_2. The approximating polynomials are

$$L_{-1}^{(2)}(P)f(z_{-1}) + L_0^{(2)}(P)f(z_0) + L_1^{(2)}(P)f(z_1)$$

for the quadratic case and

$$L_{-1}^{(3)}(P)f(z_{-1}) + L_0^{(3)}(P)f(z_0) + L_1^{(3)}(P)f(z_1) + L_2^{(3)}(P)f(z_2)$$

for the cubic case, where

$$\left\{\begin{aligned} L_{-1}^{(2)}(P) &= \frac{(-1+i)}{2}P^2 + \frac{(1-i)}{2}P \\ L_0^{(2)}(P) &= -iP^2 + (-1+i)P + 1 \\ L_1^{(2)}(P) &= \frac{(1+i)}{2}P^2 + \frac{(1-i)}{2}P \end{aligned}\right\} \qquad \left\{\begin{aligned} L_{-1}^{(3)}(P) &= \frac{(1-i)}{2}P^3 + \frac{(-3+i)}{2}P^2 + P \\ L_0^{(3)}(P) &= \frac{(1+i)}{2}P^3 - 2iP^2 - \frac{3(1-i)}{2}P + 1 \\ L_1^{(3)}(P) &= \frac{(-1+i)}{2}P^3 + \frac{(3+i)}{2}P^2 - iP \\ L_2^{(3)}(P) &= \frac{(-1-i)}{2}P^3 + iP^2 + \frac{(1-i)}{2}P \end{aligned}\right\}$$

With $P = p + iq$, we obtain for the real and imaginary parts of $L_i^{(n)}(P)$ in terms of p and q, the following expressions:

QUADRATIC

$$RL_{-1}^{(2)} = \tfrac{1}{2}(-p^2 - 2pq + q^2 + p + q), \qquad IL_{-1}^{(2)} = \tfrac{1}{2}(p^2 - q^2 - 2pq - p + q)$$

$$RL_0^{(2)} = 2pq - p - q + 1 \qquad IL_0^{(2)} = -p^2 + q^2 + p - q$$

$$RL_1^{(2)} = \tfrac{1}{2}(p^2 - q^2 - 2pq + p + q) \qquad IL_1^{(2)} = \tfrac{1}{2}(p^2 - q^2 + 2pq - p + q)$$

CUBIC

$$RL_{-1}^{(3)} = \tfrac{1}{2}(p^3 - q^3 + 3p^2q - 3pq^2 - 3p^2 + 3q^2 - 2pq + 2p)$$

$$IL_{-1}^{(3)} = \tfrac{1}{2}(-p^3 - q^3 + 3p^2q + 3pq^2 + p^2 - q^2 - 6pq + 2q)$$

$$RL_0^{(3)} = \tfrac{1}{2}(p^3 + q^3 - 3p^2q - 3pq^2 + 8pq - 3p - 3q + 2)$$

$$IL_0^{(3)} = \tfrac{1}{2}(p^3 - q^3 + 3p^2q - 3pq^2 - 4p^2 + 4q^2 + 3p - 3q)$$

$$RL_1^{(3)} = \tfrac{1}{2}(-p^3 + q^3 - 3p^2q + 3pq^2 + 3p^2 - 3q^2 - 2pq + 2q)$$
$$IL_1^{(3)} = \tfrac{1}{2}(p^3 + q^3 - 3p^2q - 3pq^2 + p^2 - q^2 + 6pq - 2p)$$
$$RL_2^{(3)} = \tfrac{1}{2}(-p^3 - q^3 + 3p^2q + 3pq^2 - 4pq + p + q)$$
$$IL_2^{(3)} = \tfrac{1}{2}(-p^3 + q^3 - 3p^2q + 3pq^2 + 2p^2 - 2q^2 - p + q)$$

The coefficients of the interpolation polynomials satisfy the following relations:

QUADRATIC

$$RL_{-1}^{(2)}((1-p)+iq) = -IL_{-1}^{(2)}(p+iq), \quad IL_0^{(2)}((1-p)+iq) = IL_0^{(2)}(p+iq),$$
$$IL_0^{(2)}(p+i(1-q)) = IL_0^{(2)}(p+iq), \quad RL_0^{(2)}((1-p)+iq) = 1 - RL_0^{(2)}(p+iq),$$
$$RL_0^{(2)}(p+i(1-q)) = 1 - RL_0^{(2)}(p+iq), \quad RL_1^{(2)}(p+i(1-q)) = IL_1^{(2)}(p+iq)$$

CUBIC

$$RL_{-1}^{(3)}((1-p)+iq) = RL_2^{(3)}(p+iq) = RL_1^{(3)}(p+i(1-q))$$
$$IL_{-1}^{(3)}((1-p)+iq) = -IL_2^{(3)}(p+iq) = IL_1^{(3)}(p+i(1-q))$$
$$RL_{-1}^{(3)}(p+i(1-q)) = RL_0^{(3)}(p+iq) = RL_1^{(3)}((1-p)+iq)$$
$$IL_{-1}^{(3)}(p+i(1-q)) = -IL_0^{(3)}(p+iq) = IL_1^{(3)}((1-p)+iq)$$

The accompanying table gives the exact values of the real and imaginary parts of the 3-point and 4-point coefficients for both p and q ranging from 0 to 1 at intervals of 0.1.

As a consequence of the relations between the cubic coefficients it would be possible to read the values of L_0, L_1 and L_2 from a table of L_{-1}. However for the sake of maximum facility in use, all the coefficients are tabulated here.

The present table is particularly useful when interpolation is to be carried out either for an isolated argument or for a large number of arguments which are irregularly scattered. Thus for 4-point interpolation for a single argument, the process of interpolation based on the present table requires a total of 4 complex multiplications (equivalent to 16 ordinary multiplications) and 2 additions whereas the method of interpolation in the real and imaginary parts each considered as a function of x and y, using real 4-point Lagrangian interpolation coefficients requires 40 multiplications and 10 additions. The table is however not an efficient tool for carrying out an extensive and systematic process of subtabulation. For such task the latter method involving the use of real interpolation coefficients is almost twice as efficient from the standpoint of the total number of multiplications and additions.

It is of interest to note (and it can be readily shown with the aid of the expressions of $L_i^{(n)}(P)$ that the value of a complex function at the center of a square in the x, y grid (i.e. $P = 0.5 + 0.5i$) obtained by the cubic interpolation is simply the average of the values at the corners of the square.

NUMERICAL ILLUSTRATIONS

A. Quadratic Interpolation

Example 1: From a 5-place table of $\tanh z$ at intervals of 0.05 in x and y[4], it is desired to obtain the value of $\tanh (0.885 + 0.265i)$. An estimate of the error in the quadratic interpolation formula based on Taylor's theorem is $(\Delta z)^3 \tanh^{(3)} z/3!$. Evaluating this expression at any of the tabular arguments close to $0.885 + 0.265i$ with the aid of $\tanh^{(3)} z = 8(\tanh z)^2 - 6(\tanh z)^4 - 2$ and using $| \Delta z | \sim 0.04$ we get approximately one unit in the fifth decimal place. In this case $z_{-1} = 0.85 + 0.30i$; $z_0 = 0.85 + 0.25i$; $z_1 = 0.90 + 0.25i$; $f(z_{-1}) = 0.72411 + 0.15454i$; $f(z_0) = 0.71390 + 0.12937i$; $f(z_1) = 0.73830 + 0.12031i$. For $P = 0.7 + 0.3i$, the accompanying table gives $L_{-1} = 0.090 - 0.210i$; $L_0 = 0.420 + 0i$; $L_1 = 0.490 + 0.210i$. We ultimately find for the expression $L_{-1}f(z_{-1}) + L_0 f(z_0) + L_1 f(z_1)$ the value $0.73396 + 0.13018i$ which agrees exactly with the value obtained by direct calculation of the function from its expression in terms of circular and hyperbolic functions.

B. Cubic Interpolation

Example 2: From an 8-place table of $\text{Log}_e z$ at intervals of 0.1 in x and y[5] it is desired to find $\text{Log}_e (3.85 + 6.18i)$. An estimate of the error in the cubic interpolation formula based on Taylor's theorem is $(\Delta z)^4 \text{Log}_e^{(4)} z/4!$. Evaluating this expression at any of the tabular arguments close to $(3.85 + 6.18i)$ and using $| \Delta z | \sim 0.09$ we obtain approximately one unit in the eighth decimal place. In this case $z_{-1} = 3.8 + 6.2i$; $z_0 = 3.8 + 6.1i$; $z_1 = 3.9 + 6.1i$; and $z_2 = 3.9 + 6.2i$. Also $f(z_{-1}) = 1.98401260 + 1.02094314i$; $f(z_0) = 1.97224510 + 1.01367194i$; $f(z_1) = 1.97964410 + 1.00194847i$; $f(z_2) = 1.99124074 + 1.0093-0266i$. For $P = 0.5 + 0.8i$ the accompanying table gives $L_{-1} = 0.3115 - 0.1335i$; $L_0 = 0.1885 - 0.0435i$; $L_1 = 0.1885 + 0.0435i$; $L_2 = 0.3115 + 0.1335i$. We ultimately find for the expression $L_{-1}f(z_{-1}) + L_0 f(z_0) + L_1 f(z_1) + L_2 f(z_2)$ the value $1.98528650 + 1.01365283i$; which agrees exactly with the value obtained by direct calculation.

Example 3: Let it be desired to calculate $\text{Log}_e (3.85 + 6.15i)$; in this case $P = 0.5 + 0.5i$. Accordingly, by the "one-quarter" rule above mentioned, the real and imaginary parts of $\text{Log}_e (3.85 + 6.15i)$ are the arithmetical averages of the real and imaginary parts of $\text{Log}_e (z)$ for the four tabular arguments $3.8 + 6.1i$; $3.9 + 6.1i$; $3.8 + 6.2i$ and $3.9 + 6.2i$. In this way we obtain the value $1.98178563 + 1.01146655i$ which agrees exactly with the value found by direct calculation.

[4] Such a table does not actually exist. The hyperbolic tangents tabulated by A. E. Kennelly in "Tables of Complex Hyperbolic and Circular Functions" are not those of the argument $z = x + iy$ but of the argument $x + i\frac{\pi}{2}y$

[5] Such a table does not actually exist.

TABLE OF COEFFICIENTS

A. *Quadratic* (3-Point Interpolation)

$$f(z_0 + Ph) = L_{-1}^{(2)}(P)f(z_{-1}) + L_0^{(2)}(P)f(z_0) + L_1^{(2)}(P)f(z_1)$$

$L_{-1}^{(2)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	.000+.000i	.045−.045i	.080−.080i	.105−.105i	.120−.120i	.125−.125i	.120−.120i	.105−.105i	.080−.080i	.045−.045i	.000+.000i
0.1	.055+.045i	.090−.010i	.115−.055i	.130−.090i	.135−.115i	.130−.130i	.115−.135i	.090−.130i	.055−.115i	.010−.090i	−.045−.055i
0.2	.120+.080i	.145+.015i	.160−.040i	.165−.085i	.160−.120i	.145−.145i	.120−.160i	.085−.165i	.040−.160i	−.015−.145i	−.080−.120i
0.3	.195+.105i	.210+.030i	.215−.035i	.210−.090i	.195−.135i	.170−.170i	.135−.195i	.090−.210i	.035−.215i	−.030−.210i	−.105−.195i
0.4	.280+.120i	.285+.035i	.280−.040i	.265−.105i	.240−.160i	.205−.205i	.160−.240i	.105−.265i	.040−.280i	−.035−.285i	−.120−.280i
0.5	.375+.125i	.370+.030i	.355−.055i	.330−.130i	.295−.195i	.250−.250i	.195−.295i	.130−.330i	.055−.355i	−.030−.370i	−.125−.375i
0.6	.480+.120i	.465+.015i	.440−.080i	.405−.165i	.360−.240i	.305−.305i	.240−.360i	.165−.405i	.080−.440i	−.015−.465i	−.120−.480i
0.7	.595+.105i	.570−.010i	.535−.115i	.490−.210i	.435−.295i	.370−.370i	.295−.435i	.210−.490i	.115−.535i	.010−.570i	−.105−.595i
0.8	.720+.080i	.685−.045i	.640−.160i	.585−.265i	.520−.360i	.445−.445i	.360−.520i	.265−.585i	.160−.640i	.045−.685i	−.080−.720i
0.9	.855+.045i	.810−.090i	.755−.215i	.690−.330i	.615−.435i	.530−.530i	.435−.615i	.330−.690i	.215−.755i	.090−.810i	−.045−.855i
1.0	1.000+.000i	.945−.145i	.880−.280i	.805−.405i	.720−.520i	.625−.625i	.520−.720i	.405−.805i	.280−.880i	.145−.945i	.000−1.000i

TABLE OF COEFFICIENTS

A. *Quadratic* (3-Point Interpolation)

$$f(z_0 + Ph) = L_{-1}^{(2)}(P)f(z_{-1}) + L_0^{(2)}(P)f(z_0) + L_1^{(2)}(P)f(z_1)$$

$L_0^{(2)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	1.00+.00i	.90+.09i	.80+.16i	.70+.21i	.60+.24i	.50+.25i	.40+.24i	.30+.21i	.20+.16i	.10+.09i	.00+.00i
0.1	.90−.09i	.82+.00i	.74+.07i	.66+.12i	.58+.15i	.50+.16i	.42+.15i	.34+.12i	.26+.07i	.18+.00i	.10−.09i
0.2	.80−.16i	.74−.07i	.68+.00i	.62+.05i	.56+.08i	.50+.09i	.44+.08i	.38+.05i	.32+.00i	.26−.07i	.20−.16i
0.3	.70−.21i	.66−.12i	.62−.05i	.58+.00i	.54+.03i	.50+.04i	.46+.03i	.42+.00i	.38−.05i	.34−.12i	.30−.21i
0.4	.60−.24i	.58−.15i	.56−.08i	.54−.03i	.52+.00i	.50+.01i	.48+.00i	.46−.03i	.44−.08i	.42−.15i	.40−.24i
0.5	.50−.25i	.50−.16i	.50−.09i	.50−.04i	.50−.01i	.50+.00i	.50−.01i	.50−.04i	.50−.09i	.50−.16i	.50−.25i
0.6	.40−.24i	.42−.15i	.44−.08i	.46−.03i	.48+.00i	.50+.01i	.52+.00i	.54−.03i	.56−.08i	.58−.15i	.60−.24i
0.7	.30−.21i	.34−.12i	.38−.05i	.42+.00i	.46+.03i	.50+.04i	.54+.03i	.58+.00i	.62−.05i	.66−.12i	.70−.21i
0.8	.20−.16i	.26−.07i	.32+.00i	.38+.05i	.44+.08i	.50+.09i	.56+.08i	.62+.05i	.68+.00i	.74−.07i	.80−.16i
0.9	.10−.09i	.15+.00i	.26+.07i	.34+.12i	.42+.15i	.50+.16i	.58+.15i	.66+.12i	.74+.07i	.82+.00i	.90−.09i
1.0	.00+.00i	.10+.09i	.20+.16i	.30+.21i	.40+.24i	.50+.25i	.60+.24i	.70+.21i	.80+.16i	.90+.09i	1.00+.00i

TABLE OF COEFFICIENTS

A. *Quadratic* (3-Point Interpolation)

$$f(z_0 + Ph) = L_{-1}^{(2)}(P)f(z_{-1}) + L_0^{(2)}(P)f(z_0) + L_1^{(2)}(P)f(z_1)$$

$L_1^{(2)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	.000+.000i	.055−.045i	.120−.080i	.195−.105i	.280−.120i	.375−.125i	.480−.120i	.595−.105i	.720−.080i	.855−.045i	1.000+.000i
0.1	.045+.045i	.090+.010i	.145−.015i	.210−.030i	.285−.035i	.370−.030i	.465−.015i	.570+.010i	.685+.045i	.810+.090i	.945+.145i
0.2	.080+.080i	.115+.055i	.160+.040i	.215+.035i	.280+.040i	.355+.055i	.440+.080i	.535+.115i	.640+.160i	.755+.215i	.880+.280i
0.3	.105+.105i	.130+.090i	.165+.085i	.210+.090i	.265+.105i	.330+.130i	.405+.165i	.490+.210i	.585+.265i	.690+.330i	.805+.405i
0.4	.120+.120i	.135+.115i	.160+.120i	.195+.135i	.240+.160i	.295+.195i	.360+.240i	.435+.295i	.520+.360i	.615+.435i	.720+.520i
0.5	.125+.125i	.130+.130i	.145+.145i	.170+.170i	.205+.205i	.250+.250i	.305+.305i	.370+.370i	.445+.445i	.530+.530i	.625+.625i
0.6	.120+.120i	.115+.135i	.120+.160i	.135+.195i	.160+.240i	.195+.295i	.240+.360i	.295+.435i	.360+.520i	.435+.615i	.520+.720i
0.7	.105+.105i	.090+.130i	.085+.165i	.090+.210i	.105+.265i	.130+.330i	.165+.405i	.210+.490i	.265+.585i	.330+.690i	.405+.805i
0.8	.080+.080i	.055+.115i	.040+.160i	.035+.215i	.040+.280i	.055+.355i	.080+.440i	.115+.535i	.160+.640i	.215+.755i	.280+.880i
0.9	.045+.045i	.010+.090i	−.015+.145i	−.030+.210i	−.035+.285i	−.030+.370i	−.015+.465i	.010+.570i	.045+.685i	.090+.810i	.145+.945i
1.0	.000+.000i	−.045+.055i	−.080+.120i	−.105+.195i	−.120+.280i	−.125+.375i	−.120+.480i	−.105+.595i	−.080+.720i	−.045+.855i	.000+1.000i

TABLE OF COEFFICIENTS

B. *Cubic* (4-Point Interpolation)

$$f(z_0 + Ph) = L_{-1}^{(3)}(P)f(z_{-1}) + L_0^{(3)}(P)f(z_0) + L_1^{(3)}(P)f(z_1) + L_2^{(3)}(P)f(z_2)$$

$L_{-1}^{(3)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	.0000+.0000i	.0855+.0045i	.1440+.0160i	.1785+.0315i	.1920+.0480i	.1875+.0625i	.1680+.0720i	.1365+.0735i	.0960+.0640i	.0495+.0405i	.0000+.0000i
0.1	.0145+.0945i	.0900+.0720i	.1415+.0595i	.1720+.0540i	.1845+.0525i	.1820+.0520i	.1675+.0495i	.1440+.0420i	.1145+.0265i	.0820+.0000i	.0495−.0405i
0.2	.0560+.1760i	.1185+.1295i	.1600+.0960i	.1835+.0725i	.1920+.0560i	.1885+.0435i	.1760+.0320i	.1575+.0185i	.1360+.0000i	.1145−.0265i	.0960−.0640i
0.3	.1215+.2415i	.1680+.1740i	.1965+.1225i	.2100+.0840i	.2115+.0555i	.2040+.0340i	.1905+.0165i	.1740+.0000i	.1575−.0185i	.1440−.0420i	.1365−.0735i
0.4	.2080+.2880i	.2355+.2025i	.2480+.1360i	.2485+.0855i	.2400+.0480i	.2255+.0205i	.2080+.0000i	.1905−.0165i	.1760−.0320i	.1675−.0495i	.1680−.0720i
0.5	.3125+.3125i	.3180+.2120i	.3115+.1335i	.2960+.0740i	.2745+.0305i	.2500+.0000i	.2255−.0205i	.2040−.0340i	.1885−.0435i	.1820−.0520i	.1875−.0625i
0.6	.4320+.3120i	.4125+.1995i	.3840+.1120i	.3495+.0465i	.3120+.0000i	.2745−.0305i	.2400−.0480i	.2115−.0555i	.1920−.0560i	.1845−.0525i	.1920−.0480i
0.7	.5635+.2835i	.5160+.1620i	.4625+.0685i	.4060+.0000i	.3495−.0465i	.2960−.0740i	.2485−.0855i	.2100−.0840i	.1835−.0725i	.1720−.0540i	.1785−.0315i
0.8	.7040+.2240i	.6255+.0965i	.5440+.0000i	.4625−.0685i	.3840−.1120i	.3115−.1335i	.2480−.1360i	.1965−.1225i	.1600−.0960i	.1415−.0595i	.1440−.0160i
0.9	.8505+.1305i	.7380+.0000i	.6255−.0965i	.5160−.1620i	.4125−.1995i	.3180−.2120i	.2355−.2025i	.1680−.1740i	.1185−.1295i	.0900−.0720i	.0855−.0045i
1.0	1.0000+.0000i	.8505−.1305i	.7040−.2240i	.5635−.2835i	.4320−.3120i	.3125−.3125i	.2080−.2880i	.1215−.2415i	.0560−.1760i	.0145−.0945i	.0000+.0000i

TABLE OF COEFFICIENTS

B. *Cubic* (4-Point Interpolation)

$$f(z_0 + Ph) = L_{-1}^{(3)}(P)f(z_{-1}) + L_0^{(3)}(P)f(z_0) + L_1^{(3)}(P)f(z_1) + L_2^{(3)}(P)f(z_2)$$

$L_0^{(3)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	1.0000+.0000i	.8505+.1305i	.7040+.2240i	.5635+.2835i	.4320+.3120i	.3125+.3125i	.2080+.2880i	.1215+.2415i	.0560+.1760i	.0145+.0945i	.0000+.0000i
0.1	.8505−.1305i	.7380+.0000i	.6255+.0965i	.5160+.1620i	.4125+.1995i	.3180+.2120i	.2355+.2025i	.1680+.1740i	.1185+.1295i	.0900+.0720i	.0855+.0045i
0.2	.7040−.2240i	.6255−.0965i	.5440+.0000i	.4625+.0685i	.3840+.1120i	.3115+.1335i	.2480+.1360i	.1965+.1225i	.1600+.0960i	.1415+.0595i	.1440+.0160i
0.3	.5635−.2835i	.5160−.1620i	.4625−.0685i	.4060+.0000i	.3495+.0465i	.2960+.0740i	.2485+.0855i	.2100+.0840i	.1835+.0725i	.1720+.0540i	.1785+.0315i
0.4	.4320−.3120i	.4125−.1995i	.3840−.1120i	.3495−.0465i	.3120+.0000i	.2745+.0305i	.2400+.0480i	.2115+.0555i	.1920+.0560i	.1845+.0525i	.1920+.0480i
0.5	.3125−.3125i	.3180−.2120i	.3115−.1335i	.2960−.0740i	.2745−.0305i	.2500+.0000i	.2255+.0205i	.2040+.0340i	.1885+.0435i	.1820+.0520i	.1875+.0625i
0.6	.2080−.2880i	.2355−.2025i	.2480−.1360i	.2485−.0855i	.2400−.0480i	.2255−.0205i	.2080+.0000i	.1905+.0165i	.1760+.0320i	.1675+.0495i	.1680+.0720i
0.7	.1215−.2415i	.1680−.1740i	.1965−.1225i	.2100−.0840i	.2115−.0555i	.2040−.0340i	.1905−.0165i	.1740+.0000i	.1575+.0185i	.1440+.0420i	.1365+.0735i
0.8	.0560−.1760i	.1185−.1295i	.1600−.0960i	.1835−.0725i	.1920−.0560i	.1885−.0435i	.1760−.0320i	.1575−.0185i	.1360+.0000i	.1145+.0265i	.0960+.0640i
0.9	.0145−.0945i	.0900−.0720i	.1415−.0595i	.1720−.0540i	.1845−.0525i	.1820−.0520i	.1675−.0495i	.1440−.0420i	.1145−.0265i	.0820+.0000i	.0495+.0405i
1.0	.0000+.0000i	.0855−.0045i	.1440−.0160i	.1785−.0315i	.1920−.0480i	.1875−.0625i	.1680−.0720i	.1365−.0735i	.0960−.0640i	.0495−.0405i	.0000+.0000i

TABLE OF COEFFICIENTS

B. *Cubic* (4-Point Interpolation)

$$f(z_0 + Ph) = L_{-1}^{(3)}(P)f(z_{-1}) + L_0^{(3)}(P)f(z_0) + L_1^{(3)}(P)f(z_1) + L_2^{(3)}(P)f(z_2)$$

$L_1^{(3)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	.0000+.0000i	.0145−.0945i	.0560−.1760i	.1215−.2415i	.2080−.2880i	.3125−.3125i	.4320−.3120i	.5635−.2835i	.7040−.2240i	.8505−.1305i	1.0000+.0000i
0.1	.0855−.0045i	.0900−.0720i	.1185−.1295i	.1680−.1740i	.2355−.2025i	.3180−.2120i	.4125−.1995i	.5160−.1620i	.6255−.0965i	.7380+.0000i	.8505+.1305i
0.2	.1440−.0160i	.1415−.0595i	.1600−.0960i	.1965−.1225i	.2480−.1360i	.3115−.1335i	.3840−.1120i	.4625−.0685i	.5440+.0000i	.6255+.0965i	.7040+.2240i
0.3	.1785−.0315i	.1720−.0540i	.1835−.0725i	.2100−.0840i	.2485−.0855i	.2960−.0740i	.3495−.0465i	.4060+.0000i	.4625+.0685i	.5160+.1620i	.5635+.2835i
0.4	.1920−.0480i	.1845−.0525i	.1920−.0560i	.2115−.0555i	.2400−.0480i	.2745−.0305i	.3120+.0000i	.3495+.0465i	.3840+.1120i	.4125+.1995i	.4320+.3120i
0.5	.1875−.0625i	.1820−.0520i	.1885−.0435i	.2040−.0340i	.2255−.0205i	.2500+.0000i	.2745+.0305i	.2960+.0740i	.3115+.1335i	.3180+.2120i	.3125+.3125i
0.6	.1680−.0720i	.1675−.0495i	.1760−.0320i	.1905−.0165i	.2080+.0000i	.2255+.0205i	.2400+.0480i	.2485+.0855i	.2480+.1360i	.2355+.2025i	.2080+.2880i
0.7	.1365−.0735i	.1440−.0420i	.1575−.0185i	.1740+.0000i	.1905+.0165i	.2040+.0340i	.2115+.0555i	.2100+.0840i	.1965+.1225i	.1680+.1740i	.1215+.2415i
0.8	.0960−.0640i	.1145−.0265i	.1360+.0000i	.1575+.0185i	.1760+.0320i	.1885+.0435i	.1920+.0560i	.1835+.0725i	.1600+.0960i	.1185+.1295i	.0560+.1760i
0.9	.0495−.0405i	.0820+.0000i	.1145+.0265i	.1440+.0420i	.1675+.0495i	.1820+.0520i	.1845+.0525i	.1720+.0540i	.1415+.0595i	.0900+.0720i	.0145+.0945i
1.0	.0000+.0000i	.0495+.0405i	.0960+.0640i	.1365+.0735i	.1680+.0720i	.1875+.0625i	.1920+.0480i	.1785+.0315i	.1440+.0160i	.0855+.0045i	.0000+.0000i

TABLE OF COEFFICIENTS

B. *Cubic* (4-Point Interpolation)

$$f(z_0 + Ph) = L_{-1}^{(3)}(P)f(z_{-1}) + L_0^{(3)}(P)f(z_0) + L_1^{(3)}(P)f(z_1) + L_2^{(3)}(P)f(z_2)$$

$L_2^{(3)}(P)$ where $P = p + iq$

q \ p	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	.0000+.0000i	.0495−.0405i	.0960−.0640i	.1365−.0735i	.1680−.0720i	.1875−.0625i	.1920−.0480i	.1785−.0315i	.1440−.0160i	.0855−.0045i	.0000+.0000i
0.1	.0495+.0405i	.0820+.0000i	.1145−.0265i	.1440−.0420i	.1675−.0495i	.1820−.0520i	.1845−.0525i	.1720−.0540i	.1415−.0595i	.0900−.0720i	.0145−.0945i
0.2	.0960+.0640i	.1145+.0265i	.1360+.0000i	.1575−.0185i	.1760−.0320i	.1885−.0435i	.1920−.0560i	.1835−.0725i	.1600−.0960i	.1185−.1295i	.0560−.1760i
0.3	.1365+.0735i	.1440+.0420i	.1575+.0185i	.1740+.0000i	.1905−.0165i	.2040−.0340i	.2115−.0555i	.2100−.0840i	.1965−.1225i	.1680−.1740i	.1215−.2415i
0.4	.1680+.0720i	.1675+.0495i	.1760+.0320i	.1905+.0165i	.2080+.0000i	.2255−.0205i	.2400−.0480i	.2485−.0855i	.2480−.1360i	.2355−.2025i	.2080−.2880i
0.5	.1875+.0625i	.1820+.0520i	.1885+.0435i	.2040+.0340i	.2255+.0205i	.2500+.0000i	.2745−.0305i	.2960−.0740i	.3115−.1335i	.3180−.2120i	.3125−.3125i
0.6	.1920+.0480i	.1845+.0525i	.1920+.0560i	.2115+.0555i	.2400+.0480i	.2745+.0305i	.3120+.0000i	.3495−.0465i	.3840−.1120i	.4125−.1995i	.4320−.3120i
0.7	.1785+.0315i	.1720+.0540i	.1835+.0725i	.2100+.0840i	.2485+.0855i	.2960+.0740i	.3495+.0465i	.4060+.0000i	.4625−.0685i	.5160−.1620i	.5635−.2835i
0.8	.1440+.0160i	.1415+.0595i	.1600+.0960i	.1965+.1225i	.2480+.1360i	.3115+.1335i	.3840+.1120i	.4625+.0685i	.5440+.0000i	.6255−.0965i	.7040−.2240i
0.9	.0855+.0045i	.0900+.0720i	.1185+.1295i	.1680+.1740i	.2355+.2025i	.3180+.2120i	.4125+.1995i	.5160+.1620i	.6255+.0965i	.7380+.0000i	.8505−.1305i
1.0	.0000+.0000i	.0145+.0945i	.0560+.1760i	.1215+.2415i	.2080+.2880i	.3125+.3125i	.4320+.3120i	.5635+.2835i	.7040+.2240i	.8505+.1305i	1.0000+.0000i

SURFACES OF MAXIMUM SHEARING STRESS

By T. Y. Thomas

At each point of an elastic medium in a state of equilibrium under the action of applied forces there are at least two local plane elements of surface having maximum shearing stress. In the general case with which we shall primarily be concerned there are exactly two such surface elements at each point and they are perpendicular. These surface elements can be considered to form two families of elements, each family containing a single element at any point of the medium and such that the unit normals to the elements of the family constitute a continuous vector field. We shall derive the conditions for the elements of such a family to generate a system of surfaces covering the medium, i.e. for the existence of surfaces such that the plane element of the family at any point is tangent to the surface passing through that point.

We assume the symmetric components $\sigma_{\alpha\beta}$ of the stress tensor to be continuous and to have continuous first partial derivatives when referred to a system of rectangular coordinates $x^\alpha (\alpha = 1, 2, 3)$. The principal directions of stress at any point are given by the vector solutions ν of the equations

$$(\sigma_{\alpha\beta} - \tau\delta_{\alpha\beta})\nu^\beta = 0, \tag{1}$$

with τ chosen to make the determinant $|\sigma_{\alpha\beta} - \tau\delta_{\alpha\beta}| = 0$. This equation has three real roots τ_1, τ_2, τ_3 and we shall treat the case where these roots are distinct (general case). We adopt the ordering $\tau_1 > \tau_2 > \tau_3$. The three unit vectors ν_1, ν_2, ν_3 which give the principal directions corresponding to these three values of τ are mutually perpendicular and are uniquely determined to within algebraic sign. Choosing the signs of the vectors ν_1, ν_2, ν_3 at an arbitrary point in any way the signs of these vectors will be determined uniquely at any other point by the requirement that each vector ν_i forms a continuous field. Finally we mention that if we select a rectangular coordinate system y with origin at any point and such that the y^1, y^2, y^3 axes lie along the vectors ν_1, ν_2, ν_3 at the point, then relative to this system the values of the components of the stress tensor at the origin will be given by $\tau_\alpha\delta_{\alpha\beta}$. These facts are well known.

It is essential for our problem that we derive a formula for the partial derivatives $\nu_{i,\mu}^\beta$ of the components ν_i^β. However, in order that our formula may be of a generality sufficient for possible use in other connections let us replace the above equations (1) by the more general equations

$$(\sigma_{\alpha\beta} - \tau g_{\alpha\beta})\nu^\beta = 0 \tag{2}$$

and allow the indices to have the range $1, \cdots, n$. Both the σ's and the g's will be assumed to be continuous and to have continuous first partial derivatives over the space under consideration. This may be taken to be the Riemann space having g as its metric tensor. The requirement that the $g_{\alpha\beta}$ are the coefficients of a positive definite quadratic differential form at each point insures the reality of the n roots τ of the determinant equation $|\sigma_{\alpha\beta} - \tau\delta_{\alpha\beta}| = 0$ and we assume

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these roots to be distinct and to have the ordering $\tau_1 > \cdots > \tau_n$. The other facts stated with reference to the equations (1) extend to the case under consideration and need not be repeated.

Since the ν_i are unit vectors and mutually perpendicular, we have

$$g_{\alpha\beta}\nu_i^\alpha \nu_j^\beta = \delta_{ij}. \tag{3}$$

The scalars τ_i are continuous and have continuous partial derivatives $\tau_{i,\mu}$ since the τ_i are simple roots of an algebraic equation whose coefficients are continuous and have continuous first partial derivatives. Let us now assume the existence of the covariant derivatives of the vectors ν_i where the covariant differentiation is based on the metric g. Then from (2) and (3) we obtain

$$(\sigma_{\alpha\beta} - \tau_i g_{\alpha\beta})\nu_{i,\mu}^\beta = (\tau_{i,\mu} g_{\alpha\beta} - \sigma_{\alpha\beta,\mu})\nu_i^\beta, \tag{4}$$

$$g_{\alpha\beta}\nu_j^\beta \nu_{i,\mu}^\alpha + g_{\alpha\beta}\nu_i^\alpha \nu_{j,\mu}^\beta = 0. \tag{5}$$

Since $|\sigma_{\alpha\beta} - \tau_i g_{\alpha\beta}| = 0$ equations (4) do not suffice for the determination of the quantities $\nu_{i,\mu}^\beta$. But there will be at most one determination of these quantities at any point P when equations (5) are adjoined to (4). To show this, select the special coordinate system with origin at P with respect to which $\sigma_{\alpha\beta} = \tau_\alpha \delta_{\alpha\beta}$ and $\nu_i^\alpha = \delta_i^\alpha$ at P. From these latter conditions and (3) we also have $g_{\alpha\beta} = \delta_{\alpha\beta}$ at the origin of this system. Then limiting our attention to those equations (5) for which $i = j$ we have[1]

$$(\sigma_{\alpha\beta} - \tau_i g_{\alpha\beta})\nu_{i,\mu}^\beta \to (\tau_\alpha \delta_{\alpha\beta} - \tau_i \delta_{\alpha\beta})\nu_{i,\mu}^\beta,$$

$$g_{\alpha\beta}\nu_i^\alpha \nu_{i,\mu}^\beta \to \sum_\beta \delta_i^\beta \nu_{i,\mu}^\beta,$$

at the origin of this system. Now consider the matrix of the coefficients of the quantities $\nu_{i,\mu}^\beta$ in the above transformed expressions. This matrix is

$$\begin{Vmatrix} (\tau_1 - \tau_i) & 0 & 0 & \cdots & 0 \\ 0 & (\tau_2 - \tau_i) & 0 & \cdots & 0 \\ 0 & 0 & (\tau_3 - \tau_i) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & (\tau_n - \tau_i) \\ \delta_i^1 & \delta_i^2 & \delta_i^3 & \cdots & \delta_i^n \end{Vmatrix}$$

For any value of i the matrix has rank n. For example, if $i = 1$ the first row consists entirely of zeros and the determinant of the remaining rows is the product $\pm(\tau_2 - \tau_1)\cdots(\tau_n - \tau_i) \neq 0$. A similar conclusion holds when $i = 2, \cdots n$. Hence the transformed equations (4) and (5) have at most one solution $\nu_{i,\mu}^\beta$ and since these equations are invariant under coordinate transformations it

[1] The summation sign will be employed whenever there is a summation on indices not in strict accordance with the convention on the basis of which a summation is to be understood when the same letter occurs once as a subscript and once as a superscript.

follows that (4) and (5) in the original coordinate system can have at most one solution $\nu^\beta_{i,\mu}$ at any point. From this fact we infer the existence of the covariant derivatives of the vectors ν_i and hence the validity of equations (4) and (5) in which the $\nu^\beta_{i,\mu}$ are the components of these covariant derivatives. Since the σ's and g's are assumed to be continuous with continuous first partial derivatives we deduce the continuity of the components $\nu^\beta_{i,\mu}$ from the above determination of these quantities as algebraic solutions of (4) and (5) and this implies the existence and continuity of the first partial derivatives of the vectors ν_i.

There remains the problem of deriving an explicit formula for the components $\nu^\beta_{i,\mu}$. For this purpose let us put

$$\nu^\beta_{i,\mu} = \sum_k \nu^\beta_k a_{ki\mu} \tag{6}$$

and substitute into (5). This gives

$$\sum_k g_{\alpha\beta}\nu^\beta_j\nu^\alpha_k a_{ki\mu} + \sum_k g_{\alpha\beta}\nu^\alpha_i\nu^\beta_k a_{kj\mu} = 0, \quad \text{or} \quad a_{ji\mu} + a_{ij\mu} = 0$$

when use is made of (3). Hence the $a_{ij\mu}$ must be skew symmetric in the first two indices. Now substituting the above expression for the $\nu^\beta_{i,\mu}$ into (4), we obtain

$$\begin{aligned}\sum_k (\sigma_{\alpha\beta} - \tau_i g_{\alpha\beta})\nu^\beta_k a_{ki\mu} &= (\tau_{i,\mu} g_{\alpha\beta} - \sigma_{\alpha\beta,\mu})\nu^\beta_i \\ &= \sum_k \tau_k g_{\alpha\beta}\nu^\beta_k a_{ki\mu} - \sum_k \tau_i g_{\alpha\beta}\nu^\beta_k a_{ki\mu} \\ &= \sum_k (\tau_k - \tau_i) g_{\alpha\beta}\nu^\beta_k a_{ki\mu}.\end{aligned}$$

Multiplying through by ν^α_m and summing on the index α gives

$$(\tau_m - \tau_i)a_{mi\mu} = \tau_{i,\mu}\delta_{im} - \sigma_{\alpha\beta,\mu}\nu^\alpha_m\nu^\beta_i.$$

Hence,

$$\begin{cases} a_{ki\mu} = -\dfrac{\sigma_{\alpha\beta,\mu}\nu^\alpha_k\nu^\beta_i}{\tau_k - \tau_i}, & (k \neq i), \\ = 0, & (k = i). \end{cases}$$

Substitution of these values of $a_{ki\mu}$ into (6) now gives

$$\nu^\beta_{i,\mu} = \sum_{k(\neq i)} \frac{\sigma_{\gamma\delta,\mu}\nu^\gamma_i\nu^\delta_k\nu^\beta_k}{\tau_i - \tau_k}, \tag{7}$$

where, for any value of i, the summation on k is over all values of k from $1, \cdots, n$ with the exception of the value $k = i$.

Formula (7) is the formula for the components $\nu^\beta_{i,\mu}$ which we set out to find. To show that this is a correct formula and that in fact we have not been misled by assuming the special form of the components $\nu^\beta_{i,\mu}$ given by (6), we have merely to substitute the expression (7) for the $\nu^\beta_{i,\mu}$ into (4) and (5). Substitution into the left member of (4) gives

$$\sum_{k(\neq i)} (\sigma_{\alpha\beta} - \tau_i g_{\alpha\beta}) \left(\frac{\sigma_{\gamma\delta,\mu} \nu_i^\gamma \nu_k^\delta \nu_k^\beta}{\tau_i - \tau_k} \right)$$

$$= \sum_{k(\neq i)} \left[\frac{\tau_k g_{\alpha\beta} \sigma_{\gamma\delta,\mu} \nu_i^\gamma \nu_k^\delta \nu_k^\beta - \tau_i g_{\alpha\beta} \sigma_{\gamma\delta,\mu} \nu_i^\gamma \nu_k^\delta \nu_k^\beta}{\tau_i - \tau_k} \right]$$

$$= - \sum_{k(\neq i)} g_{\alpha\beta} \sigma_{\gamma\delta,\mu} \nu_i^\gamma \nu_k^\delta \nu_k^\beta \tag{8}$$

$$= g_{\alpha\beta} \sigma_{\gamma\delta,\mu} \nu_i^\gamma \nu_i^\delta \nu_i^\beta - \sum_k g_{\alpha\beta} \sigma_{\gamma\delta,\mu} \nu_i^\gamma \nu_k^\delta \nu_k^\beta$$

$$= g_{\alpha\beta} \sigma_{\gamma\delta,\mu} \nu_i^\gamma \nu_i^\delta \nu_i^\beta - \sigma_{\alpha\beta,\mu} \nu_i^\beta .$$

where in obtaining the last expression we have made use of the following formula which is easily derivable from (3), namely

$$\sum_k g_{\alpha\beta} \nu_k^\beta \nu_k^\delta = \delta_\alpha^\delta .$$

We have now to show that the last expression in (8) is equal to the right member of (4). But multiplying (2), with reference to the i^{th} solution, by ν_i^α and summing on the index α we obtain $\tau_i = \sigma_{\alpha\beta} \nu_i^\alpha \nu_i^\beta$. Differentiating this relation covariantly,

$$\tau_{i,\mu} = \sigma_{\alpha\beta,\mu} \nu_i^\alpha \nu_i^\beta + 2\sigma_{\alpha\beta} \nu_i^\alpha \nu_{i,\mu}^\beta .$$

Now pass to the special coordinate system for which at the origin we have $\sigma_{\alpha\beta} = \tau_\alpha \delta_{\alpha\beta}$, $\nu_i^\alpha = \delta_i^\alpha$ and $g_{\alpha\beta} = \delta_{\alpha\beta}$. Making this transformation,

$$\sigma_{\alpha\beta} \nu_i^\alpha \nu_{i,\mu}^\beta \longrightarrow \tau_\alpha \delta_{\alpha\beta} \delta_i^\alpha \nu_{i,\mu}^\beta = \tau_i \nu_{i,\mu}^i , \tag{9}$$

$$g_{\alpha\beta} \nu_i^\alpha \nu_{i,\mu}^\beta \longrightarrow \delta_{\alpha\beta} \delta_i^\alpha \nu_{i,\mu}^\beta = \nu_{i,\mu}^i = 0, \tag{10}$$

where in the last equation in (10) there is no summation on the index i. That the expression occurring in (10) is equal to zero is of course a consequence of (5) with $i = j$. It follows from (10) that the expression in (9) vanishes and hence $\tau_{i,\mu} = \sigma_{\alpha\beta,\mu} \nu_i^\alpha \nu_i^\beta$. When this substitution for $\tau_{i,\mu}$ is made in the last expression in (8), this reduces to the right member of (4). Hence (4) is satisfied by the $\nu_{i,\mu}^\beta$ given by (7).

To show that (7) also satisfies (5) substitute into the left member of (5) to obtain

$$\sum_{k(\neq i)} \frac{g_{\alpha\beta} \sigma_{\gamma\delta,\mu} \nu_i^\gamma \nu_k^\delta \nu_k^\alpha \nu_j^\beta}{\tau_i - \tau_k} + \sum_{k(\neq j)} \frac{g_{\alpha\beta} \sigma_{\gamma\delta,\mu} \nu_j^\gamma \nu_k^\delta \nu_k^\beta}{\tau_j - \tau_k}$$

$$= \sum_{k(\neq i)} \frac{\sigma_{\gamma\delta,\mu} \nu_i^\gamma \nu_k^\delta \delta_{kj}}{\tau_i - \tau_k} + \sum_{k(\neq j)} \frac{\sigma_{\gamma\delta,\mu} \nu_j^\gamma \nu_k^\delta \delta_{ki}}{\tau_j - \tau_k} .$$

But this last expression is immediately seen to vanish if $i = j$. If $i \neq j$, for example if $i = 1$, $j = 2$, it becomes

$$\frac{\sigma_{\gamma\delta,\mu} \nu_1^\gamma \nu_2^\delta}{\tau_1 - \tau_2} + \frac{\sigma_{\gamma\delta,\mu} \nu_2^\gamma \nu_1^\delta}{\tau_2 - \tau_1} = 0.$$

Hence (5) is also satisfied by the substitution (7). Then since the quantities $\nu^{\beta}_{i,\mu}$ which satisfy (4) and (5) have been shown to have a unique determination, it follows that (7) is the correct formula[2] for the components $\nu^{\beta}_{i,\mu}$.

We now return to our original problem for which $\sigma_{\alpha\beta}$ are the components of the stress tensor referred to rectangular coordinates x^{α} ($\alpha = 1, 2, 3$) relative to which they are continuous with continuous first partial derivatives. Let v^{i}_{0} ($i = 1, 2, 3$) be the components of a vector referred to the orthogonal triad ν_1, ν_2, ν_3 at any point P (i.e. the rectangular coordinate system y with origin at P and with axes y^1, y^2, y^3 which lie along the vectors ν_1, ν_2, ν_3 at P). We assume the v_0's to be constants, i.e. the v^{i}_{0} have the same values relative to every vector triad. With respect to the fixed rectangular system x the components of this vector will not in general have constant values over the medium and will in fact be given by $v^{\alpha} = v^{i}_{0}\nu^{\alpha}_{i}$. Hence the v^{α} are continuous and have continuous first partial derivatives since these properties are enjoyed by the components ν^{α}_{i}. Since rectangular coordinates x are employed there is now no distinction between covariant and contravariant indices and so these components v^{α} can equally well be denoted by v_{α} as we shall do in certain of the following formulas.

The vector field v will admit an orthogonal system of surfaces if $\epsilon^{\alpha\beta\gamma}v_{\alpha}v_{\beta,\gamma} = 0$ where $\epsilon^{\alpha\beta\gamma}$ is defined in the usual manner, i.e. it is equal to $+1$ of α, β, γ is an even permutation of 1, 2, 3, it is equal to -1 if α, β, γ is an odd permutation of 1, 2, 3, and it is equal to zero otherwise. This condition is necessary and sufficient. But $v_{\beta,\gamma} = v^{i}_{0}\nu^{\beta}_{i,\gamma}$ and hence from (7) we have

$$(11)\qquad \sum_{\substack{\alpha,\beta,\gamma \\ i,k(\neq i)}} \frac{\epsilon^{\alpha\beta\gamma}\sigma_{k\mu,\gamma}\nu^{k}_{i}\nu^{\mu}_{k}\nu^{\beta}_{k}v^{\alpha}v^{i}_{0}}{\tau_i - \tau_k} = 0, \qquad (v^{\alpha} = v^{i}_{0}\nu^{\alpha}_{i}),$$

as the condition for the orthogonal family of surfaces. In particular we see that this family of surfaces exists when $\sigma_{\alpha\beta,\gamma} = 0$, i.e. when the components of the stress tensor are constants relative to a rectangular coordinate system. Also if $\sigma_{\alpha\beta} = 0$ for $\alpha \neq \beta$ we shall have $\tau_{\alpha} = \sigma_{\alpha\alpha}$ and $\nu^{\alpha}_{i} = \delta^{\alpha}_{i}$. Under these conditions (7) becomes

$$\nu^{\beta}_{i,\mu} = \sum_{k(\neq i)} \frac{\sigma_{ik,\mu}\delta^{\beta}_{k}}{\tau_i - \tau_k} = 0$$

and (11) is also satisfied identically. Hence whenever $\sigma_{\alpha\beta} = 0$ for $\alpha \neq \beta$ relative to a rectangular coordinate system the orthogonal family of surfaces exists.

By choosing particular values of the constants v^{i}_{0} the relation (11) becomes the condition for the generation of the surfaces of maximum shearing stress by the elements of one of the above families of elements of maximum shearing stress. When $\tau_1 > \tau_2 > \tau_3$ the two plane elements of maximum shearing stress at any point are those perpendicular planes which contain the vector ν_2 and bisect the directions determined by the vectors ν_1 and ν_3 at the point. Hence to within algebraic sign the normals to these planes have direction components relative

[2] It would be expected that this formula for the $\nu^{\beta}_{i,\mu}$ would enter in any general discussion of differential character dealing with the principal directions of a tensor.

to the triad ν_1, ν_2, ν_3 which are given by

$$v_0^1 = \frac{1}{\sqrt{2}}, \quad v_0^2 = 0, \quad v_0^3 = \frac{1}{\sqrt{2}},$$

and

$$v_0^1 = \frac{1}{\sqrt{2}}, \quad v_0^2 = 0, \quad v_0^3 = -\frac{1}{\sqrt{2}}.$$

When one of these sets of values of v_0^i is substituted into (11) *we obtain the condition for the existence of one of the families of surfaces of maximum shearing stress. Substitution of the other set of values of v_0^i gives the condition for the other family of surfaces of maximum shearing stress.* With reference to the vector triad ν_1, ν_2, ν_3 relative to which $\nu_i^n = \delta_i^n$ the explicit conditions for these families of surfaces are

$$\frac{\sigma_{12,3} - \sigma_{12,1}}{\tau_1 - \tau_2} - \frac{2\sigma_{13,2}}{\tau_1 - \tau_3} + \frac{\sigma_{23,1} - \sigma_{23,3}}{\tau_2 - \tau_3} = 0,$$

and,

$$\frac{\sigma_{12,3} + \sigma_{12,1}}{\tau_1 - \tau_2} - \frac{2\sigma_{13,2}}{\tau_1 - \tau_3} + \frac{\sigma_{23,1} + \sigma_{23,3}}{\tau_2 - \tau_3} = 0.$$

If both these conditions are imposed simultaneously they can be simplified by linear combination. Thus adding and subtracting the members of the above equations we obtain

$$\frac{\sigma_{12,3}}{\tau_1 - \tau_2} - \frac{2\sigma_{13,2}}{\tau_1 - \tau_3} + \frac{\sigma_{23,1}}{\tau_2 - \tau_3} = \frac{\sigma_{12,1}}{\tau_1 - \tau_2} + \frac{\sigma_{23,3}}{\tau_2 - \tau_3} = 0$$

as the conditions relative to the vector triads ν_1, ν_2, ν_3 for the existence of the two families of surfaces of maximum shearing stress.

When two of the quantities τ_1, τ_2, τ_3 have equal values at a point, all planes tangent to a certain cone with vertex at the point will be planes of maximum shearing stress. We shall not discuss the determination of surfaces of maximum shearing stress under this condition. However, when the values of all of the τ's are equal at each point of the medium, any plane through a point will be a plane of zero shearing stress at this point, and hence any surface (admitting a tangent plane at each point) will be a surface of zero shearing stress.

UNIVERSITY OF CALIFORNIA AT LOS ANGELES
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ON THE ASYMPTOTIC SOLUTION OF BOUNDARY VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS CONTAINING A PARAMETER

By Wolfgang Wasow

§1. Introduction

In the boundary value problems of mathematical physics the differential equations are often simplified by neglecting terms which are of higher order of differentiation than those taken into consideration. The best known example is the relationship between the theories of the flows of viscous and perfect fluids. Other examples occur in the theory of elasticity [7], [8] and in the study of non-linear vibrations.

In order to understand how the complete and the simplified problem are related one has to answer the following mathematical questions:

Given a differential equation involving a parameter ρ in such a way that the "limiting" differential equation obtained by letting ρ formally tend to infinity is of lower order,

1) Does the solution of a boundary value problem for the original differential equation converge to a limit as ρ tends to infinity, and if so, does this limit satisfy the limiting differential equation?

2) When question 1) can be answered affirmatively, which of the boundary conditions will cease to be satisfied by the limiting function?

It is clear that the limit, being a solution of a problem of lower order can in general not be expected to satisfy all the original boundary conditions. This loss of boundary conditions in the passage to the limit means that the solution (or a certain of its derivatives) does not converge uniformly everywhere. This non-uniformity of the convergence is the most interesting aspect of this type of problem for the physicist. In fluid dynamics it leads to the formation of boundary layers. In the boundary layer theory the questions 1) and 2) formulated above are usually answered with the help of physical considerations, because the differential equations are mostly so complicated that a purely mathematical approach is at present out of the question.

The chief aim of this paper is to show that for homogeneous linear ordinary differential equations depending linearly on the parameter the questions 1) and 2) can be answered by a simple and easily applicable rule. This rule will be derived from an asymptotic representation of the solution which is of interest in itself (formula (31)).

§2. The Asymptotic Fundamental System

The differential equation to be considered is

$$N(y) + \rho M(y) = 0 \qquad (1)$$

where ρ is a positive parameter, y a real function of the real variable x, and

$$N(y) \equiv \sum_{\nu=0}^{n} a_\nu(x) y^{(n-\nu)}(x)$$
$$M(y) \equiv \sum_{\mu=0}^{m} b_\mu(x) y^{(m-\mu)}(x) \qquad (2)^1$$

At the endpoints of the interval $\alpha \leq x \leq \beta$ the n boundary conditions

$$L_i(y) = l_i \qquad (i = 1, 2, \cdots, n) \tag{3}$$

(the l_i are constants) are imposed, where the expressions $L_i(y)$ are of the form

$$L_i(y) = \begin{cases} y^{(\lambda_i)}(\beta), & \text{for } i = 1, 2, \cdots, r \\ y^{(\tau_i)}(\alpha), & \text{for } i = r+1, r+2, \cdots, n. \end{cases} \tag{4}$$

The boundary conditions are arranged in such a way that

$$n > \lambda_1 > \lambda_2 > \cdots > \lambda_r \geq 0, \qquad n > \tau_{r+1} > \tau_{r+2} > \cdots > \tau_n \geq 0. \tag{5}$$

The functions $a_\nu(x)$, $b_\mu(x)$ are assumed to have at least n bounded derivatives in the interval $\alpha \leq x \leq \beta$. If, as will be supposed, $n > m$, the limiting differential equation $M(y) = 0$ is of lower order than the differential equation (1). The essential assumptions

$$a_0(x) = 1, \quad b_0(x) \neq 0, \quad \text{everywhere in the interval } \alpha \leq x \leq \beta,$$

exclude singularities. The problem thus defined will be denoted by $(\mathcal{P})$.

To facilitate the formulation of asymptotic equalities the notation $[f(x)]$ will be used for expressions of the form

$$[f(x)] = f(x) + \frac{E(x, \rho)}{\rho^a}$$

where $a > 0$, and $E(x, \rho)$ is uniformly bounded in x for $\rho \geq R > 0$ (R is a constant).

To the differential equation (1) the methods for asymptotic integration developed by Noaillon [2] and Turrittin [4] can be applied.[2] A straightforward calculation, omitted here, leads to the following theorem.

THEOREM. *The differential equation* (1) *has n linearly independent solutions of the form*

$$U_\nu(x, \rho) = \begin{cases} \exp\left(\sigma \int_a^x \varphi_\nu(\xi)\, d\xi\right) \cdot [\eta(x)], & (\nu = 1, 2, \cdots, n-m) \quad (6) \\ [u_{\nu-n+m}(x)], & (\nu = n-m+1, n-m+2, \cdots, n) \quad (7) \end{cases}$$

Here the following abbreviations are used:

1) $\sigma = \sqrt[n-m]{\rho}$ *(The real value of the root is meant.)*

[1] We use the abbreviation $y^{(k)}$ for $\dfrac{d^k y}{dx^k}$.

[2] The theories of Birkhoff [1] and Tamarkin [3] cannot be applied here because the "characteristic equation" belonging to (1) has a multiple root.

2) *$\varphi_1(x), \varphi_2(x), \cdots, \varphi_{n-m}(x)$ are the $n - m$ values of $(-b_0(x))^{1/(n-m)}$ arranged in such a way that*

$$\text{Re } (\varphi_1) \geq \text{Re } (\varphi_2) \geq \cdots \geq \text{Re } (\varphi_{n-m}) \tag{8}$$

4) $$\eta(x) = \exp\left(\int_a^x \frac{a_1(\xi)b_0(\xi) - b_1(\xi)}{n-m}\, d\xi\right)$$

5) *The functions $u_\mu(x)$, $(\mu = 1, 2, \cdots, m)$ are m linearly independent solutions of the differential equation $M(y) = 0$.*

It also follows from Nosaillon's theory that the equalities (6) and (7) may be formally differentiated at least $n - 1$ times, i.e.,

$$U_\nu^{(i)}(x, \rho) = \begin{cases} \sigma^i \exp\left(\int_a^x \varphi_\nu(\xi)\, d\xi\right)\cdot[\varphi_\nu^i(x)\cdot\eta(x)], & (\nu = 1, 2, \cdots, n-m), \quad (9) \\ [u_{\nu-n+m}^{(i)}(x), & (\nu = n-m+1, \cdots, n). \quad (10) \end{cases}$$

REMARKS: The functions $\varphi_\nu(x)$ are the $(n - m)^{\text{th}}$ roots of a real function and are therefore proportional to the $(n - m)^{\text{th}}$ roots of unity. More precisely: Set

$$k(x) = \begin{cases} |\, b_0^{1/(n-m)}\,|, & \text{if } b_0 < 0 \\ |\, b_0^{1/(n-m)}\,|\, e^{\pi i/(n-m)}, & \text{if } b_0 > 0 \end{cases}$$

and let

$$\epsilon = e^{2\pi i/(n-m)},$$

then the functions $\varphi_\nu(x)$ are—in different order—equal to

$$k,\ k\epsilon,\ k\epsilon^2, \cdots,\ k\epsilon^{n-m-1}.$$

Note also that in (8) the "$>$" and "$=$" signs alternate, because the sequence of the $\varphi_\nu(x)$ consists of pairs of conjugate numbers.

§3. Formulation of the Main Theorem for Odd Values of $n - m$

Let us at first assume that $n - m$ is odd. The case when $n - m$ is even will be discussed in §5. For the formulation of our Main Theorem it is convenient to introduce integers p and q defined by the following table.

If $n - m \equiv 1 \pmod 4$, $b_0 > 0$ or $n - m \equiv 3 \pmod 4$, $b_0 < 0$ then $q = \dfrac{n-m+1}{2}$, $p = \dfrac{n-m-1}{2}$

If $n - m \equiv 1 \pmod 4$, $b_0 < 0$ or $n - m \equiv 3 \pmod 4$, $b_0 > 0$ then $q = \dfrac{n-m-1}{2}$, $p = \dfrac{n-m+1}{2}$

If, for the problem $(\mathcal{P})$

$$p \leq r \quad \text{and} \quad q \leq n - r \tag{11}$$

then $(\mathcal{P})$ will be called *regular*. Remember that r and $n - r$ are the numbers of boundary conditions belonging to the right and left end-point, respectively.

If $(\mathcal{P})$ is regular we denote by (p) the new problem characterized by the differential equation $M(y) = 0$ and the m boundary conditions obtained from $(\mathcal{P})$ by canceling the first p boundary conditions at $x = \beta$ and the first q boundary conditions at $x = \alpha$, starting from those containing the highest derivatives. This cancellation rule becomes meaningless if $(\mathcal{P})$ is not regular.

Main Theorem for Odd Values of $n - m$: Denote by $U(x, \rho)$ the solution of a regular problem $(\mathcal{P})$ and by $u(x)$ the solution of the corresponding problem (p). Then

$$\lim_{\rho \to \infty} U(x, \rho) = u(x), \quad \text{in} \quad \alpha < x < \beta \text{ [3]} \tag{12}$$

provided

1° *the homogeneous problem corresponding to (p) has only the solution $y(x) \equiv 0$,*

2° *No two of the boundary conditions canceled at $x = \alpha$ involve orders of differentiation that are congruent modulo $n - m$, and the same is true at $x = \beta$.*

REMARK: If $(\mathcal{P})$ is not regular, or if one of the conditions 1° and 2° is not satisfied it can be shown that $U(x, \rho)$ will "in general" not converge. Here we shall content ourselves with indicating simple examples where $U(x, \rho)$ is divergent, because one of these conditions is not satisfied.

EXAMPLES. 1. Non-regular problem: $y^{(4)} + \rho y' = 0$,

$$y''(0) = y'(0) = y(0) = 0, \qquad y(1) = 1.$$

2. Condition 1° not satisfied: $y'' - \rho y' = 0$,

$$y'(0) = 1, \qquad y(1) = 0.$$

3. Condition 2° not satisfied: $y^{(4)} - \rho y' = 0$,

$$y'''(0) = y(0) = 0, \quad y'(1) = 1, \quad y(1) = 0.$$

An elementary (though somewhat lengthy) explicit calculation shows that the solutions of these three problems diverge as $\rho \to \infty$.

§4. Proof of the Main Theorem for Odd Values of $n - m$

For all values of ρ that are not eigen values of problem $(\mathcal{P})$ this problem has a solution $U(x, \rho)$ which can be written in the form

$$U(x, \rho) = \sum_{\nu=1}^{n} c_\nu(\rho) U_\nu(x, \rho) \tag{13}$$

where the functions $c_\nu(\rho)$ are solutions of the system of linear equations

$$\sum_{\nu=1}^{n} c_\nu(\rho) L_i(U_\nu) = l_i, \qquad (i = 1, 2, \cdots, n). \tag{14}$$

For $L_i(U_\nu)$ equations (4) and (9) yield, when $\nu = 1, 2, \cdots, n - m$,

$$L_i(U_\nu) = \begin{cases} \sigma^{\lambda_i} \exp(\sigma w_\nu) \cdot [\varphi_\nu^{\lambda_i}(\beta) \cdot \eta(\beta)], & (i = 1, 2, \cdots, r) \\ \sigma^{\tau_i} [\varphi_\nu^{\tau_i}(\alpha) \cdot \eta(\alpha)], & (i = r + 1, r + 2, \cdots, n) \end{cases} \tag{15}$$

[3] The endpoints of the interval can be included except when the last boundary condition canceled at that endpoint has order of differentiation zero.

where

$$w_\nu = \int_a^\beta \varphi_\nu(\xi)\cdot d\xi \tag{16}$$

while for $\nu = n - m + 1, \cdots, n$ one has, on account of (10),

$$L_i(U_\nu) = [L_i(u_{\nu-n+m})]. \tag{17}$$

In order to find the $c_\nu(\rho)$ of (13) from (14) one has to calculate the determinant

$$\Delta(\rho) = \text{Det } \{L_i(U)\}, \qquad (i,\ \nu = 1, 2, \cdots, n) \tag{18}$$

and the determinants $\Delta_\nu(\rho)$, $(\nu = 1, 2, \cdots, n)$ obtained by replacing the terms in the ν-th column of $\Delta(\rho)$ by the l_i. Then

$$U(x, \rho) = \sum_{\nu=1}^{n} \frac{\Delta_\nu(\rho)}{\Delta(\rho)}\, U_\nu(x, \rho),$$

valid for all values of ρ for which $\Delta(\rho) \neq 0$.

Observe that the inequalities (8) imply similar inequalities for the numbers w_ν, $(\nu = 1, 2, \cdots, n - m)$, i.e.

$$\text{Re } (w_1) \geq \text{Re } (w_2) \geq \cdots \geq \text{Re } (w_{n-m}). \tag{20}$$

If the values (15) and (17) for the $L_i(U_\nu)$ are substituted into (18) it is seen that the matrix formed by the last m columns of this determinant can be written in the form

$$([L_i(u_\kappa)]), \qquad \begin{matrix}(i = 1, 2, \cdots, n)\\ (\kappa = 1, 2, \cdots, m)\end{matrix} \tag{21}$$

All the minors of this matrix tend to finite limits as $\rho \to \infty$. The asymptotic value of $\Delta(\rho)$ will now be found by expanding $\Delta(\rho)$ in terms of its $n - m$ first columns and by investigating the order of magnitude of the minors in this expansion.

Among these minors the one of highest order of magnitude is obtained by taking the rows

$$1, 2, \cdots, p, \qquad r+1, r+2, \cdots, r+q.$$

To show this we first use (15) and write this minor in the form

$$D(\rho) = \eta^p(\beta)\eta^q(\alpha)\sigma^T \begin{vmatrix} \exp(\sigma w_1)\cdot[\varphi_1^{\lambda_1}(\beta)] & \cdots & \exp(\sigma w_{n-m})[\varphi_{n-m}^{\lambda_1}(\beta)] \\ \cdots\cdots & & \cdots\cdots \\ \exp(\sigma w_1)\cdot[\varphi_1^{\lambda_p}(\beta)] & \cdots & \exp(\sigma w_{n-m})[\varphi_{n-m}^{\lambda_p}(\beta)] \\ [\varphi_1^{\tau_{r+1}}(\alpha)] & \cdots & [\varphi_{n-m}^{\tau_{r+1}}(\alpha)] \\ \cdots\cdots & & \cdots\cdots \\ [\varphi_1^{\tau_{r+q}}(\alpha)] & \cdots & [\varphi_{n-m}^{\tau_{r+q}}(\alpha)] \end{vmatrix} \tag{22}$$

where

$$T = \sum_{\nu=1}^{p} \lambda_\nu + \sum_{\mu=r+1}^{r+q} \tau_\mu.$$

From (16) and the definition of p at the beginning of §3 it is readily seen that Re (w_i) is positive for $i \leq p$ and negative for $i > p$. Hence, if the determinant in (22) is expanded with respect to the minors of its first p rows, the exponential function of greatest order of magnitude is the one occurring in the minor formed by the first p rows and columns. This shows that

$$D(\rho) = [\eta^p(\beta)\eta^q(\alpha)K_\beta K_\alpha]\sigma^T \exp\left(\sigma \sum_{\nu=1}^{p} w_\nu\right) \tag{23}$$

with

$$\begin{aligned} |K_\beta| &= |\operatorname{Det}(\varphi_i^{\lambda_\kappa}(\beta))|, \qquad (i,\kappa = 1,2,\cdots,p) \\ |K_\alpha| &= |\operatorname{Det}(\varphi_{p+i}^{\tau_\kappa+\varepsilon}(\alpha))|, \quad (i,\kappa = 1,2,\cdots,q) \end{aligned} \tag{24}$$

provided $K_\beta \not\equiv 0$ and $K_\alpha \not\equiv 0$. In order to show that the other minors of the first $n - m$ columns of $\Delta(\rho)$ are all of lower order of magnitude, consider first the minors for which the number h of rows taken from the first r rows of $\Delta(\rho)$ is different from p. If such a minor is expanded in terms of these h rows it is seen that the highest exponential factor of every term has a smaller real part than that in (23), because either not all w_i with positive real parts are present $(h < p)$, or some w_i with negative real part are included $(h > p)$. If $h = p$, but the p rows are not the rows $1, 2, \cdots, p$ of $\Delta(\rho)$, then the exponential factor of the minor will be the same as in (23), but the exponent of σ will be less than T since any other p of the λ_ν's have a sum less than $\sum_{\nu=1}^{p} \lambda_\nu$. The same reasoning accounts for the choice of the remaining $n - m - p$ rows of the minor.

To prove that $K_\beta \not\equiv 0$ note that by the remark at the end of §2 and the definition of p the numbers $\varphi_1(\beta), \varphi_2(\beta), \cdots, \varphi_p(\beta)$ are those determinations of $(-b_0)^{1/(n-m)}$ which lie in the right half-plane and are therefore, in different order, equal to the numbers

$$k(\beta)\epsilon^{t+1}, k(\beta)\epsilon^{t+2}, \cdots, k(\beta)\epsilon^{t+p} \tag{25}$$

where t is a certain integer, which is determined modulo $n - m$ only. Substituting the expressions (25) into (24) the relation

$$|K_\beta| = k^{\sum_{\nu=1}^{p} \lambda_\nu}(\beta) \cdot \operatorname{Det}(\epsilon^{(t+i)\lambda_\kappa}) \qquad (i,\kappa = 1, 2, \cdots, p) \tag{26}$$

is obtained. Let $\zeta_\kappa = \epsilon^{\lambda_\kappa}$, then (26) becomes

$$|K_\beta| = |k^{\sum_{\nu=1}^{p} \lambda_\nu}(\beta) \cdot \operatorname{Det}(\zeta_\kappa^{t+i})| = |k^{\sum_{\nu=1}^{p} \lambda_\nu}(\beta) \cdot (\zeta_1\zeta_2\cdots\zeta_p)^t \cdot V(\zeta_1, \zeta_2, \cdots, \zeta_p)|$$

where $V(\zeta_1, \zeta_2, \cdots, \zeta_p)$ is the Vandermonde determinant. Hence, K_β is zero if and only if some of the numbers ζ_ν are equal. But $\zeta_\kappa = \zeta_l$ means $\lambda_\kappa \equiv \lambda_l \pmod{n - m}$, which contradicts hypothesis 2° of the Main Theorem. Thus, $K_\beta \not\equiv 0$ is proved. The proof of $K_\alpha \not\equiv 0$ is strictly analogous. Hence $D(\rho)$ is actually the minor of $\Delta(\rho)$ of highest order of magnitude.

The cofactor of the minor $D(\rho)$ in $\Delta(\rho)$ is—except for a possible minus sign—that minor of the matrix (21) which is formed by the rows of $\Delta(\rho)$ not contained in $D(\rho)$. This cofactor can be written in the form $[\delta]$ where

$$\delta = \pm \mathrm{Det}\,(L_\nu(u_\mu)), \quad \begin{matrix} \nu = p+1, p+2, \cdots, r, & r+q+1, \cdots, n \\ \mu = 1, 2, \cdots, m \end{matrix} \tag{27}$$

Observe that the boundary terms occurring in the determinant in (27) are precisely those belonging to the problem (P) defined in §3. Assumption 1° of the Main Theorem implies therefore $\delta \not\equiv 0$.
Hence $\Delta(\rho) = D(\rho)[\delta]$, or

$$\Delta(\rho) = [K \cdot \delta]\sigma^T \exp\left(\sigma \sum_{\omega=1}^{p} w_\omega\right), \tag{28}$$

where K is a non-vanishing constant.

The method used for the calculation of $\Delta(\rho)$ can also be applied to the determinants $\Delta_\nu(\rho)$. For $\nu > n - m$ the determinant $\Delta_\nu(\rho)$ is distinguished from $\Delta(\rho)$ only in one of the last m columns. Therefore all the considerations used in the evaluation of $\Delta(\rho)$ remain valid for $\Delta_\nu(\rho)$, if the determinant δ is replaced by the determinant δ_ν obtained by substituting the corresponding l_i's for the elements of the $\nu - (n-m)^{\text{th}}$ column of δ. Thus it follows that

$$\Delta_\nu(\rho) = [K \cdot \delta_\nu]\sigma^T \exp\left(\sigma \sum_{\omega=1}^{p} w_\omega\right), \quad \text{for} \quad \nu > n - m,$$

hence

$$\frac{\Delta_\nu(\rho)}{\Delta(\rho)} = \left[\frac{\delta_\nu}{\delta}\right], \quad \text{for} \quad \nu > n - m \tag{29}$$

For $\nu \leq n - m$ the ν^{th} column of $\Delta_\nu(\rho)$—i.e. the one consisting of the l_i's—can be placed behind the others without changing the absolute value of $\Delta_\nu(\rho)$. Then the method used for the asymptotic evaluation of $\Delta(\rho)$ can be applied to $\Delta_\nu(\rho)$, when m is replaced by $m' = m + 1$ and w_ν is omitted from the sequence $w_1, w_2, \cdots, w_{n-m}$. Instead of (29) the following formula is obtained, for $\nu \leq n - m$:

$$\frac{\Delta_\nu(\rho)}{\Delta(\rho)} = \begin{cases} \sigma^{-\lambda_\nu} \exp(-\sigma w_\nu)[Q_\nu], & \text{for} \quad \nu \leq p \\ \sigma^{-\tau_\nu + q}[Q_\nu], & \text{for} \quad p < \nu \leq n - m \end{cases} \tag{30}$$

The Q_ν are constants. To prove (30) for $\nu \leq p$, observe that here w_ν,—the term to be omitted from the sequence $w_1, w_2, \cdots, w_{n-m}$—has a positive real part. Hence the asymptotic formula (28) can be applied to $\Delta_\nu(\rho)$ as well as to $\Delta(\rho)$, if p and m are replaced by $p' = p - 1$, $m' = m + 1$. This leads to

$$\Delta_\nu(\rho) = [K^{(\nu)}\delta_\nu]\sigma^{T_\nu} \exp\left(\sigma\left(\sum_{\omega=1}^{p} w_\omega - w_\nu\right)\right)$$

where $T_\nu = \sum_{\mu=r+1}^{r+q} \tau_\mu + \sum_{\omega=1}^{p-1} \lambda_\omega$, and $K^{(\nu)}$ is a constant. Division of this formula

for $\Delta_\nu(\rho)$ by (28) proves the first part of (30). For $p < \nu \leq n - m$ the reasoning is different only in that here $p' = p$, $q' = q - 1$ (and, of course, $m' = m + 1$).

Substituting into (19) the values of $\Delta_\nu(\rho)/\Delta(\rho)$ from (29) and (30) and replacing $U_\nu(x, \rho)$ by its values as given in (6) and (7) one finds the asymptotic representation

$$U(x, \rho) = [u(x)] + \sigma^{-\lambda} \sum_{\nu=1}^{p} \exp\left(\sigma \int_\beta^x \varphi_\nu(\xi)\, d\xi\right) \cdot [Q_\nu \eta(x)] + \sigma^{-\tau} \sum_{\nu=p+1}^{n-m} \exp\left(\sigma \int_\alpha^x \varphi_\nu(\xi)\, d\xi\right) \cdot [Q_\nu \eta(x)], \tag{31}$$

where $\lambda = \lambda_p$ and $\tau = \tau_{r+q}$ are the lowest orders of differentiation occurring in the canceled boundary conditions, and

$$u(x) = \sum_{\mu=1}^{m} \frac{\delta_{n-m+\mu}}{\delta} u_\mu(x).$$

$u(x)$ is obviously the solution of problem (p), because of the definitions of δ and δ_ν. (31) is the asymptotic representation referred to in the introduction. Now

$$\operatorname{Re}\left(\int_\beta^x \varphi_\nu(\xi)\, d\xi\right) < 0 \quad \text{in} \quad \alpha \leq x < \beta, \quad \text{for} \quad \nu \leq p$$

$$\operatorname{Re}\left(\int_\alpha^x \varphi_\nu(\xi)\, d\xi\right) < 0 \quad \text{in} \quad \alpha < x \leq \beta, \quad \text{for} \quad \nu > p$$

on account of the definition of p. Therefore (12) follows from (31). This proves the Main Theorem for odd values of $n - m$.

§5. The Main Theorem for Even Values of $n - m$

If $n - m$ is even the numbers p and q have to be defined as follows:

$$\left.\begin{array}{l}\text{If } n - m \equiv 0 \pmod 4,\ b_0 > 0\\ \text{or } n - m \equiv 2 \pmod 4,\ b_0 < 0\end{array}\right\} \quad \text{then} \quad q = \frac{n-m}{2}, \quad p = \frac{n-m}{2} \tag{32}$$

$$\left.\begin{array}{l}\text{If } n - m \equiv 0 \pmod 4,\ b_0 < 0\\ \text{or } n - m \equiv 2 \pmod 4,\ b_0 > 0\end{array}\right\} \quad \text{then} \quad q = \frac{n-m-2}{2},\ p = \frac{n-m-2}{2} \tag{33}$$

For problems ($\mathcal{P}$) of the type (32) the formulation and the proof of the Main Theorem are literally the same as in §§3 and 4.

For problems ($\mathcal{P}$) of the type (33), which may be called the *exceptional type*, the following changes in the statement of the Main Theorem are necessary:

a) In order to obtain problem (p) from ($\mathcal{P}$) one cancels first $p + q$ $(= n - m - 2)$ boundary conditions as in §3, but, in addition, also the two boundary conditions of highest order of differentiation from the remaining ones. Problem ($\mathcal{P}$) will be called *regular*, if this cancellation rule is meaningful, i.e., if (11) holds and the last two boundary conditions to be canceled are *uniquely* defined.

b) In addition to assumptions 1°, 2° of §3 the two conditions below must be satisfied:

3° The last two boundary conditions canceled in accordance with a) belong to the *same* endpoint.

4° $m \neq 0$.

Then (12) holds, as in §3.

REMARK: The conditions 3° and 4° are necessary in the sense that if one of them is not satisfied $U(x, \rho)$ may diverge, as is shown by the following simple examples.

EXAMPLE 1. (3° not satisfied) $y''' + \rho y' = 0, y'(0) = 0, y(0) = 1, y'(1) = 0$.

EXAMPLE 2. ($m = 0$) $y'' + \rho y = 0, y(0) = 1, y'(0) = 0$.

PROOF OF THE MAIN THEOREM FOR PROBLEMS OF THE EXCEPTIONAL TYPE: The modifications of the theorem in this case are made necessary by the fact that the real parts of $\varphi_{p+1}(x)$ and $\varphi_{p+2}(x) = -\varphi_{p+1}(x)$ are now zero. A reasoning similar to that used for odd values of $n - m$ shows that in this case the minor $D^*(\rho)$ of greatest order of magnitude among all minors of the first $n - m$ columns of $\Delta(\rho)$ must contain the $n - m - 2$ rows

$$1, 2, \cdots, p, \quad r+1, r+2, \cdots, r+q-2. \tag{34}$$

Since $\operatorname{Re} w_{(p+1)} = \operatorname{Re} (w_{p+2}) = 0$ the remaining two rows to be chosen do not influence the order of the exponential factor in $D^*(\rho)$ but only the exponent T^* of the power of σ occurring as a factor in $D^*(\rho)$. T^* is the sum of all the λ_ν and τ_μ occurring in the rows that form $D^*(\rho)$. Hence the two remaining rows to be chosen are those corresponding to the boundary conditions containing the highest orders of differentiation, excluding the rows already contained in the sequence (34). These are exactly those boundary conditions which do not belong to (P). Thus one obtains, instead of (23),

$$D^*(\rho) = [K^*]\sigma^{T^*} \exp\left(\sigma \sum_{\nu=1}^{h} w_\nu\right) \tag{35}$$

where $K^* \neq 0$ and

$$T^* = \sum_{\nu=1}^{h} \lambda_\nu + \sum_{\mu=r+1}^{r+n-m-h} \tau_\mu, \tag{36}$$

h being equal to p or $p + 2$ as the case may be. The possibility $h = p + 1$ is excluded by assumption 3°. This exclusion is necessary, for if 3° does not hold then the expansion of the determinant $D^*(\rho)$ with respect to its first $p + 1$ rows yields, instead of (35) the more complicated expression

$$D^*(\rho) = \left\{[K^*] \exp\left(\sigma \sum_{\nu=1}^{p+1} w_\nu\right) + [K'^*] \exp\left(\sigma \sum_{\nu=1}^{p} w_\nu + \sigma w_{p+2}\right)\right\} \sigma^{T^*},$$

which is composed of *two* terms of the same order of magnitude. It may then happen that $D^*(\rho)$ and hence $\Delta(\rho)$ vanish for an unbounded sequence of positive values of ρ (as in example 1 of § 3) and $U(x, \rho)$ will not converge.

Proceeding as in §4 one obtains finally

$$\frac{\Delta_\nu(\rho)}{\Delta(\rho)}\,U_\nu(x,\rho) = \begin{cases} [Q_\nu^*\,\eta(x)]\,\sigma^{-\lambda_h}\exp\left(\sigma\int_\beta^x \varphi_\nu(\xi)\,d\xi\right), & (\nu = 1, 2, \cdots, p) \quad (36) \\ [Q_\nu^*\,\eta(x)]\,\sigma^{-s}\exp\left(\sigma\int_\gamma^x \varphi_\nu(\xi)\,d\xi\right), & (\nu = p+1, p+2) \quad (37) \\ [Q_\nu^*\,\eta(x)]\,\sigma^{-r_\nu+m-n-\lambda}\exp\left(\sigma\int_\alpha^x \varphi_\nu(\xi)\,d\xi\right), & (38) \\ & (\nu = p+3, \cdots, n-m) \end{cases}$$

where the Q_ν^* are constants, $\gamma = \alpha$ or $\gamma = \beta$, according as $h = p$ or $h = p + 2$, and s is the order of differentiation of the last boundary condition canceled in accordance with the cancellation rule formulated at the beginning of this §. No new viewpoints occur in the proof of (36) and (38), but formula (37) requires a special discussion. It will suffice to treat, e.g., the case when $\nu = p + 1$, $h = p + 2$. First $\Delta(\rho)_{p+1}$ has to be evaluated asymptotically. We assume again that the $(p+1)^{st}$ column of $\Delta_{p+1}(\rho)$—whose elements are the l_i—has already been placed behind all the other columns. We calculate $\Delta_{p+1}(\rho)$ by expanding it with respect to the minors of its first $n - m - 1$ columns. The minor of greatest order of magnitude in this expansion consists of the $n - m - 1$ rows $1, 2, \cdots, p, p+1, r+1, \cdots, r+m-n-p-2$ and its asymptotic expression is therefore

$$D_{p+1}^*(\rho) = [K^{(p+1)*}]\,\sigma^{r^*-\lambda_{p+2}}\exp\left(\sigma\left(\sum_{\omega=1}^{p+2} w_\omega - w_{p+1}\right)\right).$$

Using this relation together with (35) and (6) and noting that the cofactors of $D^*(\rho)$ and $D_{p+1}^*(\rho)$ are bounded functions of ρ, we obtain (37) for $\nu = p + 1$ with $s = \lambda_{p+2}$, $\gamma = \beta$. When $\nu = p + 2$ or $h = p$ the proof is similar. If $s \neq 0$ all the right hand members of (37) tend to zero. $s = 0$ is excluded by the conditions 3° and 4° and the assumption that $(\mathcal{P})$ is a regular problem.

The convergence of $U(x, \rho)$ to $u(x)$ follows then as in §4.

§6. Additional Results

1. More general boundary conditions. Consider, instead of the boundary conditions (3), (4) §2 the more general ones below,

$$L_i(y) \equiv \begin{cases} \sum_{\mu=1}^{n} \beta_{i\mu}\,y^{(\mu-1)}(\beta) = l_i & (i = 1, 2, \cdots, r) \\ \sum_{\mu=1}^{n} \alpha_{i\mu}\,y^{(\mu-1)}(\alpha) = l_i & (i = r+1, r+2, \cdots, n), \end{cases} \quad (39)$$

where the constants $\alpha_{i\mu}$, $\beta_{i\mu}$ and l_i are only restricted by the conditions that the n boundary conditons are to be independent and compatible.

If in (39) the $L_i(y)$ and the l_i are subjected to the same linear transformation with constant coefficients and non-vanishing determinant then the resulting equations constitute a set of boundary conditions equivalent to (39) in the sense that a function satisfying (39) satisfies also the new boundary conditions, and

vice versa. An elementary algebraic consideration shows that (39) can, in particular, be replaced by equivalent boundary conditions of the form

$$L_i^*(y) = \begin{cases} \sum_{\mu=1}^{\lambda_{i+1}} \beta_{i\mu}^* y^{(\mu-1)}(\beta) = l_i^*, & (i = 1, 2, \cdots, r) \\ \sum_{\nu=1}^{\tau_{i+1}} \alpha_{i\nu}^* y^{(\nu-1)}(\alpha) = l_i^*, & (i = r+1, r+2, \cdots, n) \end{cases} \tag{40}$$

where the λ_i and τ_i are numbers satisfying the inequalities (5) of §2, and $\beta_{i,\lambda_i+1}^* = \alpha_{i,\tau_i+1}^* = 1$.

The asymptotic equations (15) and (17) remain obviously valid if the L_i are replaced by the L_i^*, and hence the whole asymptotic calculation is applicable to the L_i^* without any change whatsoever. This proves the

THEOREM: *Denote by* $(\mathcal{P}^*)$ *the problem defined by the differential equation* (1) *and the boundary conditions* (39). *Replace these boundary conditions by the equivalent ones* (40) *and define a problem* $(\mathfrak{p}^*)$, *starting from these boundary conditions, in literal analogy with the definition of problem* $(\mathfrak{p})$ *in* §§3 *and* 5. *Then the Main Theorem remains literally valid for problem* $(\mathcal{P}^*)$.

2. The non-homogeneous differential equation $N(y) + \rho M(y) = \rho f(x)$. It may be mentioned without proof that the statement of the Main Theorem of §3 can be proved to remain valid for the corresponding non-homogeneous equation, when $n - m = 1$, provided none of the boundary conditions of problem $(\mathfrak{p})$ involves an order of differentiation higher than $m - 1$. To prove this statement one expresses the solution of the non-homogeneous problem in the usual way as an integral containing Green's function and uses the asymptotic fundamental system (6) and (7) for an asymptotic evaluation of this integral.

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ON THE THEORY OF BENDING OF ELASTIC PLATES

By Eric Reissner

Introduction

In this paper there is established a system of differential equations of the sixth order for the linear problem of bending of thin plates. The form of these equations is such that results obtained by their application coincide with the results of the classical theory of thin plates except for narrow edge zones. On the basis of the present equations it is possible and necessary to satisfy three boundary conditions along the edges of a plate while, as is well known, the classical theory leads to two boundary conditions only. The history and significance of the problem has recently been discussed by J. J. Stoker.[1]

Formulation of the problem

We consider an elastic body bounded by two parallel planes and by a cylindrical surface perpendicular to the two planes. The distance h between the two parallel planes is assumed to be small compared with the remaining linear dimensions of the body, which because of this order of magnitude relation may be called a "plate." A coordinate system (x, y, z) is chosen such that the faces of the plate are the planes $z = \pm h/2$. The cylindrical boundary of the plate may be given by equations of the form $x = x(s)$, $y = y(s)$ where s stands for the circumferential arc length. It is assumed that the two faces of the plate are free of shear stress while the normal traction σ_z is a given function of x and y. The resultant of the surface stresses σ_z is balanced by stresses distributed over the cylindrical boundary of the plate.

We begin by assuming that the bending stresses are distributed linearly over the thickness of the plate,

$$\sigma_x = \frac{M_x}{h^2/6}\frac{z}{h/2}, \qquad \sigma_y = \frac{M_y}{h^2/6}\frac{z}{h/2}, \qquad \tau_{xy} = \frac{H}{h^2/6}\frac{z}{h/2} \tag{1a, b, c}$$

By means of the differential equations of equilibrium we obtain as expressions for the transverse shear stresses

$$\tau_{xz} = \frac{V_x}{2h/3}\left(1 - \left(\frac{z}{h/2}\right)^2\right), \qquad \tau_{yz} = \frac{V_y}{2h/3}\left(1 - \left(\frac{z}{h/2}\right)^2\right) \tag{2a, b}$$

The shear stress resultants V are given in terms of the stress couples M and H,

$$V_x = \frac{\partial M_x}{\partial x} + \frac{\partial H}{\partial y}, \qquad V_y = \frac{\partial H}{\partial x} + \frac{\partial M_y}{\partial y} \tag{3a, b}$$

For the remaining normal stress component there results

$$\sigma_z = -\frac{3}{4}\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y}\right)\left(\frac{z}{h/2} - \frac{1}{3}\left(\frac{z}{h/2}\right)^3\right) \tag{4}$$

[1] *Mathematical problems connected with the bending and buckling of elastic plates.* Bulletin of the American Mathematical Society, 48, pp. 247–261 (1942).

if it is assumed that the vertical loads p are distributed over both faces of the plate as follows

$$\sigma_z\left(x, y, \pm\frac{h}{2}\right) = \pm\frac{p}{2} \tag{5}$$[2]

Comparison of equations (5) and (4) gives

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = -p \tag{6}$$

Substituting equations (3) in equation (6) we have

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 H}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p \tag{7}$$

To obtain further relations for the three stress couples it is necessary to make use of the stress strain relations. This can be done in various ways. For the present purpose it is convenient to do this by means of Castigliano's Theorem of Least Work. For simplicity's sake it may be assumed that there are prescribed along the cylindrical portion of the boundary either the values of σ_n, and or τ_{ns}, and or τ_{nz} (in such a way that equations (1) and (2) remain saisfied) or vanishing of the work of the stresses σ_n and or τ_{ns} and or τ_{nz}. The Theorem of Least Work then states that among all statically correct states of stress the state of stress which also satisfies the stress strain relations and the displacement boundary conditions is characterized by the condition that the variation of the strain energy vanishes.

Taking isotropic materials the strain energy is given by

$$\Pi = \frac{1}{2E}\iiint \{\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu(\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z) + 2(1+\nu)(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2)\}\, dx\, dy\, dz \tag{8}$$

Substituting the values of the stresses from equations (1), (2) and (4) the integration with respect to z may be carried out. With

$$\left.\begin{aligned} \int_{-h/2}^{h/2}\left(\frac{z}{h/2}\right)^2 dz &= \frac{h}{3}, \qquad \int_{-h/2}^{h/2}\left(1-\left(\frac{z}{h/2}\right)^2\right)^2 dz = \frac{8h}{15} \\ \int_{-h/2}^{h/2}\frac{z}{h/2}\left(\frac{z}{h/2} - \frac{1}{3}\left(\frac{z}{h/2}\right)^3\right) dz &= \frac{4h}{15} \end{aligned}\right\} \tag{9}$$

there results

$$\Pi = \frac{1}{2E}\iint \left\{\frac{12}{h^3}[M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1+\nu)H^2] + \frac{12(1+\nu)}{5h}\left[\left(\frac{\partial M_x}{\partial x} + \frac{\partial H}{\partial y}\right)^2 + \left(\frac{\partial M_y}{\partial y} + \frac{\partial H}{\partial x}\right)^2\right] - \frac{12\nu}{5h}p(M_x + M_y) + \int_{-h/2}^{h/2}\sigma_z^2\, dz\right\} dx\, dy \tag{10}$$

[2] The difference between this condition and the condition that the surface loads are applied to one face of the plate only is not important for the present purposes.

The variation of Π according to equation (10) is to be made equal to zero in such a way that the equilibrium equation (7) remains satisfied. Introducing according to the rules of the calculus of variations a Lagrangian multiplier $w(x, y)$ we then have

$$\delta \iint \left\{ \frac{12}{h^3} [M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1+\nu)H^2] \right.$$
$$+ \frac{12(1+\nu)}{5h} \left[\left(\frac{\partial M_x}{\partial x} + \frac{\partial H}{\partial y} \right)^2 + \left(\frac{\partial M_y}{\partial y} + \frac{\partial H}{\partial x} \right)^2 \right]$$
$$- \frac{12\nu}{5h} p(M_x + M_y) + \int_{-h/2}^{h/2} \sigma_z^2 \, dz$$
$$\left. + 2Ew(x, y) \left[\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 H}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p \right] \right\} dx \, dy = 0 \quad (11)$$

Equation (11) is the basic relationship from which every other result will be deduced. It may be remarked that the results of the classical theory of plate bending are obtained if in equation (11) one neglects the strain energy of the transverse shear and normal stresses. The main point of the present work is recognition of the fact that application of the variational principle without this neglection leads to a system of equations for which three conditions can and have to be satisfied along the edges of the plate. This is due to the fact that retention of the transverse shear stress terms increases the order of the resultant system of differential equations.

The differential equations and boundary conditions of the theory

Carrying out the variations in equation (11),

$$\iint \left\{ \frac{12}{h^3} [(M_x - \nu M_y)\delta M_x + (M_y - \nu M_x)\delta M_y + 2(1+\nu)H\delta H] \right.$$
$$+ \frac{12(1+\nu)}{5h} \left[\left(\frac{\partial M_x}{\partial x} + \frac{\partial H}{\partial y} \right) \delta \left(\frac{\partial M_x}{\partial x} + \frac{\partial H}{\partial y} \right) + \left(\frac{\partial H}{\partial x} + \frac{\partial M_y}{\partial y} \right) \delta \left(\frac{\partial H}{\partial x} + \frac{\partial M_y}{\partial y} \right) \right]$$
$$\left. - \frac{9\nu}{10h} p\delta(M_x + M_y) + Ew\delta \left(\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 H}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} \right) \right\} dx \, dy = 0 \quad (12)$$

Integrating by parts and rearranging,

$$\iint \left\{ \left[\frac{12}{h^3} (M_x - \nu M_y) - \frac{12(1+\nu)}{5h} \frac{\partial V_x}{\partial x} - \frac{12\nu}{10h} p + E \frac{\partial^2 w}{\partial x^2} \right] \delta M_x \right.$$
$$+ \left[\frac{12}{h^3} (M_y - \nu M_x) - \frac{12(1+\nu)}{5h} \frac{\partial V_y}{\partial y} - \frac{12\nu}{10h} p + E \frac{\partial^2 w}{\partial y^2} \right] \delta M_y$$
$$+ \left[\frac{12}{h^3} 2(1+\nu)H - \frac{12(1+\nu)}{5h} \left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x} \right) + 2E \frac{\partial^2 w}{\partial x \partial y} \right] \delta H \Big\} dx \, dy$$
$$+ \oint \left\{ \left[\frac{12(1+\nu)}{5h} V_n - E \frac{\partial w}{\partial n} \right] \delta M_n + \left[\frac{12(1+\nu)}{5h} V_s - E \frac{\partial w}{\partial s} \right] \delta H_{ns} \right.$$
$$\left. + [Ew]\delta V_n \right\} ds = 0 \quad (13)$$

In equation (13) the line integral is taken along the cylindrical boundary of the plate, s and n referring to the tangential and normal direction, respectively. It is seen that the variational equation (13) implies three differential equations and three boundary conditions. The first two differential equations may be solved for M_x and M_y. If this is done there follow, with the notation,

$$\frac{Eh^3}{12(1-\nu^2)} = D \tag{14}$$

as differential equations,

$$M_x - \frac{h^2}{5(1-\nu)}\left(\frac{\partial V_x}{\partial x} + \nu\frac{\partial V_y}{\partial y}\right) - \frac{h^2\nu}{10(1-\nu)}p = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \tag{15a}$$

$$M_y - \frac{h^2}{5(1-\nu)}\left(\frac{\partial V_y}{\partial y} + \nu\frac{\partial V_x}{\partial x}\right) - \frac{h^2\nu}{10(1-\nu)}p = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \tag{15b}$$

$$H - \frac{h^2}{10}\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right) = -(1-\nu)D\frac{\partial^2 w}{\partial x \partial y} \tag{15c}$$

and as boundary conditions

$$\left.\begin{aligned} M_n &= \bar{M}_n \quad \text{or} \quad \frac{\partial w}{\partial n} = \frac{12(1+\nu)}{5hE}V_n \\ H_{ns} &= \bar{H}_{ns} \quad \text{or} \quad \frac{\partial w}{\partial s} = \frac{12(1+\nu)}{5hE}V_s \\ V_n &= \bar{V}_n \quad \text{or} \quad w = 0 \end{aligned}\right\} \tag{16a, b, c}$$

In addition to the differential equations (15) a fourth differential equation is the equilibrium condition (7).

Before showing that the system of equations (15) and (7) is indeed of the sixth order, the significance of the three displacement boundary conditions may be explained. The second parts of equations (16) express the fact that the plate is supported in such a manner that the edge moments and forces can do no work. For this to be so it is necessary that the linear elements perpendicular to the middle surface of the plate do not change direction when the plate is stressed. Because of the effect of shear deformation this is not equivalent to the condition of no change of slope of the middle surface of the plate. Equation (16a) states what the change of slope of middle surface has to be in order that M_n may do no work. Equation (16b) states that when $w = 0$ along the boundary the distribution of edge twisting moments has to be such that $\bar{V}_s = 0$.

Transformation of the differential equations (7) and (15)

Differentiating equation (15a) twice with respect to x, equation (15b) twice with respect to y and equation (15c) twice with respect to x and y there follows by addition, because of equation (7),

$$-p - \frac{h^2}{5(1-\nu)}\Delta\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y}\right) - \frac{\nu h^2}{10(1-\nu)}\Delta p = -D\Delta\Delta w$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Observing equation (6) we obtain as differential equation for the deflection w,

$$D\Delta\Delta w = p - \frac{2-\nu}{10(1-\nu)} h^2 \Delta p \tag{17}$$

Clearly, the second term on the right of equation (17) is insignificant unless p changes appreciably within distances of the order of the plate thickness.

As the deflection w satisfies a fourth order equation as in the classical theory the additional two orders must be contained in equations (15). Adding equations (15a) and (15b) there follows, in view of equation (6),

$$M_x + M_y + \frac{h^2}{5(1-\nu)} p = -(1+\nu) D\Delta w \tag{18}$$

With the help of equations (18) and (7) one may transform equations (15) into three equations each one of which contains only one of the three quantities M_x, M_y and H. Taking first equation (15a) in the form

$$M_x - \frac{h^2}{5(1-\nu)}\left(\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 H}{\partial x \partial y} + \nu \frac{\partial^2 M_y}{\partial y^2} + \nu \frac{\partial^2 H}{\partial x \partial y}\right) - \frac{\nu h^2}{10(1-\nu)} p = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$$

and substituting $\frac{\partial^2 H}{\partial x \partial y}$ from equation (7)

$$M_x - \frac{h^2}{5(1-\nu)}\left[\frac{\partial^2 M_x}{\partial x^2} + \nu \frac{\partial^2 M_y}{\partial y^2} + \frac{1+\nu}{2}\left(p - \frac{\partial^2 M_x}{\partial x^2} - \frac{\partial^2 M_y}{\partial y^2}\right)\right] - \frac{\nu h^2}{10(1-\nu)} p = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$$

Taking now M_y from equation (18) and rearranging there results

$$M_x - \frac{h^2}{10}\Delta M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) + \frac{h^2}{10(1-\nu)}\left[\nu p + \frac{h^2}{5}\frac{\partial^2 p}{\partial y^2} + (1-\nu^2) D \frac{\partial^2 \Delta w}{\partial y^2}\right] \tag{19a}$$

A corresponding equation is obtained for M_y.

The equation involving H only follows from equation (15c) which may be written

$$H - \frac{h^2}{10}\left[\Delta H + \frac{\partial^2 (M_x + M_y)}{\partial x \partial y}\right] = -(1-\nu) D \frac{\partial^2 w}{\partial x \partial y}$$

Substituting $M_x + M_y$ from equation (18) there follows

$$H - \frac{h^2}{10}\Delta H = -(1-\nu)D\frac{\partial^2 w}{\partial x \partial y} - \frac{h^2}{10}\left[\frac{h^2}{5(1-\nu)}\frac{\partial^2 p}{\partial x \partial y} + (1+\nu)D\frac{\partial^2 \Delta w}{\partial x \partial y}\right] \quad (19c)$$

The three equations (19) together with equations (18), (17) and (7) may for the present be considered as the final form of the system of differential equations governing the problem.

The "edge effect" equations of plate theory

For the analysis of the edge effect under consideration various terms in the differential equations are unessential. Omitting these terms there remains as the relevant system of equations

$$D\Delta\Delta w = p \quad (20)$$

$$M_x + M_y = -(1+\nu)D\Delta w \quad (21)$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 H}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = p \quad (7)$$

$$M_x - \frac{h^2}{10}\Delta M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \quad (22)$$

$$M_y - \frac{h^2}{10}\Delta M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \quad (23)$$

$$H - \frac{h^2}{10}\Delta H = -(1-\nu)D\frac{\partial^2 w}{\partial x \partial y} \quad (24)$$

The edge effect terms in these equations are those having h^2 as a factor. If they are omitted the customary equations of plate theory remain.

The integration of the above system of equations is effected by first finding w from (20), by substituting the result in (22) to (24) whence M_x, M_y and H are obtained as combinations of particular solutions and complementary functions, $M_x = M_{x,\text{hom.}} + M_{x,\text{part.}}$, and so on. Equations (20) and (7) are two equations connecting the three complementary functions,

$$M_{x,\text{hom.}} + M_{y,\text{hom.}} = 0 \quad (25)$$

$$\frac{\partial^2 M_{x,\text{hom.}}}{\partial x^2} + 2\frac{\partial^2 H_{\text{hom.}}}{\partial x \partial y} + \frac{\partial^2 M_{y,\text{hom.}}}{\partial y^2} = 0 \quad (26)$$

As an illustration consider a plate of the shape of a semi-infinite rectangular strip ($0 \leq y \leq a, 0 \leq x$). Assume that the edges $y = 0, a$ are simply supported

and that the edge $x = 0$ is acted upon by a distribution of bending and twisting moments and of vertical shear force in the following manner.

$$x = 0, \quad \left\{ \begin{aligned} M_x &= M_0 \sin \frac{n\pi}{a} y \\ H &= H_0 \cos \frac{n\pi}{a} y \\ V_x &= V_0 \sin \frac{n\pi}{a} y \end{aligned} \right\} \tag{27a, b, c}$$

From equation (20) there is obtained as a suitable expression for w,

$$w = \sin \frac{n\pi}{a} y \left(A + B \frac{n\pi}{a} x \right) e^{-(n\pi/a)y} \tag{28}$$

As we assume

$$h \ll na \tag{29}$$

we deduce from equations (22) to (24), except for negligibly small terms,

$$\left. \begin{aligned} M_x &= C_1 e^{-\sqrt{10}\, x/h} \sin \frac{n\pi}{a} y - D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \\ M_y &= C_2 e^{-\sqrt{10}\, x/h} \sin \frac{n\pi}{a} y - D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \\ H &= C_3 e^{-\sqrt{10}\, x/h} \cos \frac{n\pi}{a} y - (1 - \nu) D \frac{\partial^2 w}{\partial x \partial y} \end{aligned} \right\} \tag{30a, b, c}$$

From equation (25) follows

$$C_1 + C_2 = 0 \tag{31}$$

From equation (26) follows

$$\frac{10}{h^2} C_1 + 2 \frac{\sqrt{10}}{h} \frac{n\pi}{a} C_3 - \left(\frac{n\pi}{a} \right)^2 C_2 = 0 \tag{32}$$

whence in view of equation (29)

$$C_2 = -C_1, \qquad C_3 = -\frac{\sqrt{10}}{2n\pi} \frac{a}{h} C_1 \tag{33a, b}$$

Thus, there are three constants of integration by means of which the three boundary conditions (27) may be satisfied.

Connection with the theory of moderately thick plates[3]

Considering in equations (19) the terms with h^2 as small correction terms one might approximate these small terms by substituting in them the relations be-

[3] See A. E. H. Love, *The Mathematical Theory of Elasticity*, Cambridge, 1934, pp. 465–487.

tween M_x, M_y, H and w which hold if the correction terms are omitted. If this is done there follows

$$\left.\begin{aligned} M_x &= -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) + \frac{h^2}{5} D\frac{\partial^2 \Delta w}{\partial y^2} - \frac{1-2\nu}{1-\nu}\frac{h^2}{10}p \\ M_y &= -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) + \frac{h^2}{5} D\frac{\partial^2 \Delta w}{\partial x^2} - \frac{1-2\nu}{1-\nu}\frac{h^2}{10}p \\ H &= -(1-\nu)D\frac{\partial^2 w}{\partial x \partial y} - \frac{h^2}{5} D\frac{\partial^2 \Delta w}{\partial x \partial y} \end{aligned}\right\} \quad (31a, b, c)$$

Corresponding formulas may be found in Love's Treatise on page 473 when there is no surface load p. The only difference between Love's formulas and the present formulas is a factor $\left(1 + \frac{\nu}{8}\right)$ in the small fourth derivative terms.[4]

The important difference between equations (31) and (19) is that by the simplification of equations (19) to equations (31) the sixth order problem is reduced to a fourth order problem with resultant loss of the possibility to satisfy the three boundary conditions of the problem.

It may be remarked that the approach to the theory of moderately thick plates and to more general problems which has been initiated by G. D. Birkhoff[5] and which consists in series developments of the solutions in terms of a thickness parameter does not appear to be suitable for the analysis of the edge effect[6] which is the main concern of the present paper. It may be of considerable interest to determine the inner reason for this difference between the variational approach as employed here and the approach by way of the thickness-parameter series solutions.

In conclusion it may be stated that it is entirely possible by means of the variational method to obtain more accurate solutions than the one here obtained. One may for instance use instead of equations (1) more general equations of the form

$$\sigma_x = \frac{M_x}{h^2/6}\frac{z}{h/2} + \mu_x\left[\frac{z}{h/2} - \frac{5}{3}\left(\frac{z}{h/2}\right)^3\right] \quad (32a)$$

where now M_x and μ_x and the corresponding quantities occurring in σ_y and τ_{xy} are to be determined by application of Castigliano's theorem. Instead of the two-term expressions of equation (32) one can, in principle, also use suitable n-term expressions. It is probable, however, that the calculations necessary for such extensions of the present results soon become quite involved.

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[4] Note that in Love the plate thickness is $2h$ while here it is h.

[5] *Circular Plates of Variable Thickness*, Phil. Mag. (Ser. 6), 43, pp. 953-962 (1922).

[6] J. N. Goodier, *On the problems of the beam and the plate in the theory of elasticity*. Trans. Roy. Soc., Canada, 33, Sect. 3 (1938).

NOTE ON THE THEOREM OF THE SYMMETRY OF THE STRESS TENSOR[1]

By Eric Reissner

It is known in the Analysis of Stress that the stress tensor is not symmetrical when "body moments" (as in the case of magneto striction) are present. The purpose of this note is to show, that the absence of body moments in itself is not sufficient to insure symmetry of the state of stress at a point. By means of a simple example it will be shown that *the proof for the symmetry of the stress tensor* ($\tau_{xy} = \tau_{yx}, \cdots$) *involves the condition that the rates of change of the components of stress remain finite.*

An example illustrating this statement is given by the state of plane strain in a rectangular wedge region, with one edge of the region loaded by a uniform surface shear stress $\tau_{xy} = \tau_0$ and the other edge free of load (Fig. 1). Thus a state of stress is produced where for the corner element $\tau_{xy} \neq \tau_{yx}$. The analysis will show that, however small the corner element, the variation of the stresses over its faces remains the same and the eccentricity of the stress resultants balances the moment of the shear stress τ_0.

The explicit solution of this problem is obtained by known methods, taking for Airy's stress function which has to satisfy the equation $\nabla^2\nabla^2 F = 0$ the following expression,

$$F = \sigma_0 r^2 \left(\frac{\pi}{8} \cos 2\theta - \frac{1}{4} \sin 2\theta - \frac{\pi}{8} + \frac{1}{2}\theta \right) \tag{1}$$

This satisfies all the boundary conditions, namely

$$\theta = 0 : \sigma_\theta \equiv \frac{\partial^2 F}{\partial r^2} = 0, \qquad \tau \equiv -\frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial F}{\partial \theta} \right) = 0 \tag{2}$$

$$\theta = \frac{\pi}{2} : \sigma_\theta = 0, \qquad \tau = \tau_0 \tag{3}$$

Writing this solution in cartesian coordinates,

$$F = \frac{\tau_0}{2} \left[\frac{\pi}{4} (x^2 - y^2) - xy - \frac{\pi}{4} (x^2 + y^2) + (x^2 + y^2) \tan^{-1}\left(\frac{y}{x} \right) \right] \tag{4}$$

the stresses in the x- and y-directions have the values

$$\sigma_x \equiv \frac{\partial^2 F}{\partial y^2} = \tau_0 \left[-\frac{\pi}{2} + \frac{xy}{x^2 + y^2} + \tan^{-1}\left(\frac{y}{x} \right) \right] \tag{5}$$

$$\sigma_y \equiv \frac{\partial^2 F}{\partial x^2} = \tau_0 \left[-\frac{xy}{x^2 + y^2} + \tan^{-1}\left(\frac{y}{x} \right) \right] \tag{6}$$

[1] This note contains the result of a discussion between Professor H. Reissner and the present writer.

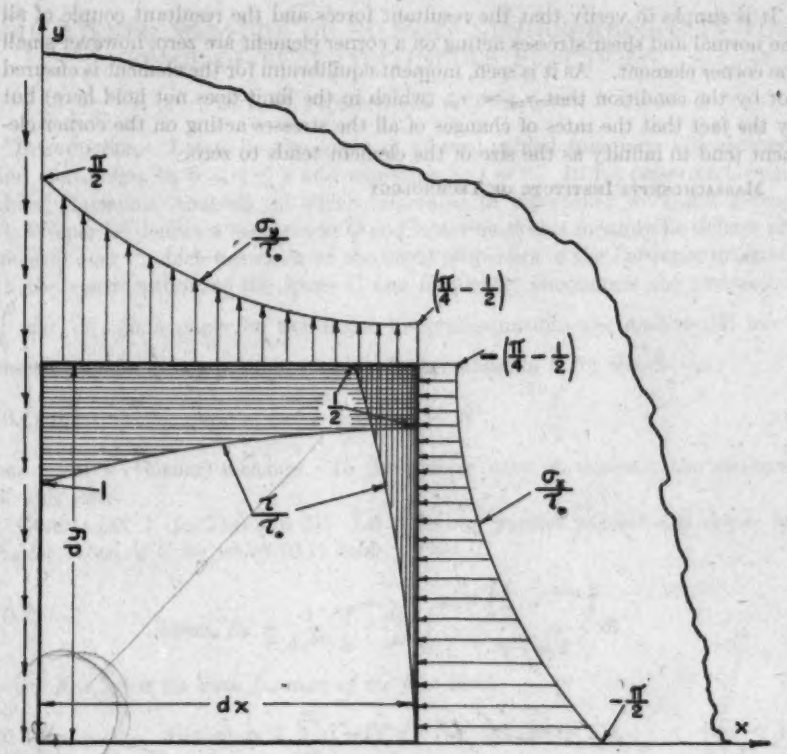

FIG. 1

$$\tau = -\frac{\partial^2 F}{\partial x\,\partial y} = \tau_0 \frac{y^2}{x^2+y^2} \tag{7}$$

In order to show that the stress distribution is as sketched in Fig. 1 there is obtained from (5) to (7)

$$y = x; \quad \sigma_y = \left(\frac{\pi}{4}-\frac{1}{2}\right)\tau_0, \qquad \sigma_x = \left(\frac{1}{2}-\frac{\pi}{4}\right)\tau_0, \qquad \tau = \tfrac{1}{2}\tau_0 \tag{8}$$

$$y = 0, \qquad x \neq 0; \quad \sigma_x = -\frac{\pi}{2}\tau_0, \qquad \tau = 0 \tag{9}$$

$$y \neq 0, \qquad x = 0; \quad \sigma_y = \frac{\pi}{2}\tau_0, \qquad \tau = \tau_0 \tag{10}$$

It is simple to verify that the resultant forces and the resultant couple of all the normal and shear stresses acting on a corner element are zero, however small the corner element. As it is seen, moment equilibrium for the element is ensured not by the condition that $\tau_{xy} = \tau_{yx}$ (which in the limit does not hold here) but by the fact that the rates of changes of all the stresses acting on the corner element tend to infinity as the size of the element tends to zero.

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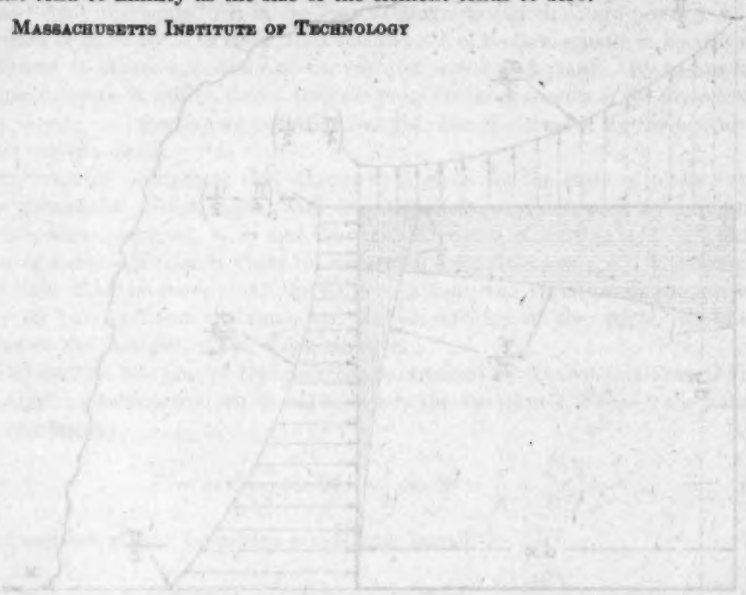

THE WIENER MEASURE OF HILBERT NEIGHBORHOODS IN THE SPACE OF REAL CONTINUOUS FUNCTIONS

By R. H. Cameron and W. T. Martin

Introduction. Let C be the space of all real-valued functions $x(t)$ defined and continuous on $0 \leq t \leq 1$ and vanishing at $t = 0$. In his paper on Generalized Harmonic Analysis (in which references to his earlier work are given) N. Wiener [1] defines a measure in C and in terms of this measure he defines an integral over C which has many of the usual properties of the Lebesgue integral. In the consideration of the space C one frequently encounters the expression $\int_0^1 x^2(t)\,dt$. In a paper on non-linear integral equations the authors [2] have shown that for every positive number R the subset of C for which

$$\int_0^1 [x(t)]^2\,dt < R^2 \tag{0.1}$$

has positive (Wiener) measure. In the present note we evaluate the measure of this set:

Corollary 1 (to Theorem I). *Let R be any positive number and denote by S_R the subset of C for which* (0.1) *holds. Then*

$$\text{meas}_w S_R = \frac{1}{4\pi^{3/2}} \int_0^{R^2} \frac{du}{u^{3/2}} \int_0^{\pi/2} \frac{\vartheta_1'\left(\frac{t}{2}, e^{-1/4u}\right)}{\sqrt{\cos t}}\,dt, \tag{0.2}$$

where $\vartheta_1(z, q)$ is the theta function of the first kind,

$$\vartheta_1(z, q) = 2\sum_{n=0}^{\infty} (-1)^n q^{(2n+1)^2/4} \sin(2n+1)z, \qquad |q| < 1, \tag{0.3}$$

and

$$\vartheta_1'(z, q) = \frac{\partial}{\partial z}\vartheta_1(z, q). \tag{0.4}$$

Moreover, for R in the neighborhood of the origin

$$\text{meas}_w S_R \sim \sqrt{\frac{2}{\pi}}\, e^{-1/16R^2}[2^2R - 2^5R^3 + 1\cdot3\cdot2^8R^5 - 1\cdot3\cdot5\cdot2^{11}R^7 + \cdots]. \tag{0.5}$$

In a recent paper [3] the authors have shown that

$$\int_C e^{-i\lambda\int_0^1 x^2(t)\,dt}\,d_w x = \sqrt{\sec\sqrt{-i\lambda}}, \qquad (-\infty < \lambda < \infty), \tag{0.6}$$

where $\sqrt{\sec\sqrt{-i\lambda}}$ is to have the determination which is positive when λ is purely imaginary and contained in the interval $(0, i\pi^2/4)$. By use of (0.6) and the theory of Fourier transforms we evaluate the expression

$$\int_C F\left[\int_0^1 x^2(t)\,dt\right] d_w x \tag{0.7}$$

for a general class of functions $F(u)$ defined on $0 \leq u < \infty$. The result (0.2) will be obtained by specializing $F(u)$ to be $F(u) = 1$ on $0 \leq u < R^2$, $F(u) = 0$ elsewhere.

As was remarked at the beginning of this section, all the functions $x(\cdot)$ of C vanish at the origin. With only slight modifications we can handle functionals of

$$\int_0^1 [x(t) + \xi]^2 \, dt$$

where ξ is an arbitrary real number. If ξ has Gaussian distribution with variance unity we can integrate over the product space $[x(\cdot) \in C, -\infty < \xi < \infty]$ or, for example, over the portion of the product space for which $\int_0^1 [x(t) + \xi]^2 dt < R^2$. This enables us to obtain in particular a measure for the set of continuous functions $\bar{x}(t)$ for which

$$\int_0^1 [\bar{x}(t)]^2 \, dt < R^2$$

where it is no longer assumed that $\bar{x}(0) = 0$.

In the next section we state the main results and in succeeding sections we give the proofs.

1. Statement of main results.

THEOREM 1. *Let $F(u)$ be a (real or complex) measurable function defined on $0 < u < \infty$. Then a necessary and sufficient condition that*

$$F\left[\int_0^1 x^2(t) \, dt\right] \tag{1.1}$$

be Wiener summable over C is that

$$\frac{F(u)}{u^{3/2}} \int_0^{\pi/2} \frac{\vartheta_1'\left(\frac{t}{2}, e^{-1/4u}\right)}{\sqrt{\cos t}} \, dt \tag{1.2}$$

be of class L_1 on $0 < u < \infty$. Moreover, if this condition is satisfied,

$$\int_C F\left[\int_0^1 x^2(t) \, dt\right] d_w x = \frac{1}{4\pi^{3/2}} \int_0^\infty \frac{F(u)}{u^{3/2}} \int_0^{\pi/2} \frac{\vartheta_1'\left(\frac{t}{2}, e^{-1/4u}\right)}{\sqrt{\cos t}} \, dt. \tag{1.3}$$

(In (1.2) and (1.3) $\vartheta_1(z, q)$ is the theta function of the first kind (see (0.3)) and $\vartheta_1'(z, q)$ is the partial derivative with respect to z).

Relation (0.2) of Corollary 1 follows at once from this result. Relation (0.5) will be derived in Section 9.

THEOREM 2. *Let ξ be an arbitrary real number and let $F(u)$ be a (real or complex)*

measurable function defined on $0 < u < \infty$. *Then a necessary and sufficient condition that*

$$F\left[\int_0^1 \{x(t) + \xi\}^2\, dt\right] \tag{1.4}$$

be Wiener summable over C is that the function

$$F(u)H(-u, \xi)$$

be of class L_1 *on* $0 < u < \infty$, *where* $H(u, \xi)$ *is the Fourier transform of the function*

$$h(\lambda, \xi) = \sqrt{\sec\sqrt{-i\lambda}}\, e^{\xi^2\sqrt{-i\lambda}\tan\sqrt{-i\lambda}}, \tag{1.5}$$

that is,

$$H(u, \xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(\lambda, \xi)e^{-i\lambda u}\, d\lambda. \tag{1.6}$$

Moreover, if $F(u)H(-u, \xi)$ *belongs to* $L_1(0, \infty)$, *then*

$$\int_C F\left[\int_0^1 \{x(t) + \xi\}^2\, dt\right] d_w x = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} F(u)H(-u, \xi)\, du. \tag{1.7}$$

REMARK. In Lemma 4 (below) it will be shown that the function $h(\lambda, \xi)$ of (1.5) belongs to $L(-\infty < \lambda < \infty)$ and hence the integral in (1.6) exists as an absolutely convergent integral and not just as a limit in the mean.

The proof of Theorem 2 will be based upon the following five lemmas, the first of which generalizes the result (0.6) mentioned in the introduction and the second of which is given for example by Titchmarsh [4, p. 54].

LEMMA 1.

$$\int_C e^{-i\lambda\int_0^1 [x(t)+\xi]^2\, dt}\, d_w x = \sqrt{\sec\sqrt{-i\lambda}}\, e^{\xi^2\sqrt{-i\lambda}\tan\sqrt{-i\lambda}}, \tag{1.8}$$

$$\left(\begin{matrix} -\infty < \lambda < \infty \\ -\infty < \xi < \infty \end{matrix}\right),$$

where $\sqrt{\sec\sqrt{-i\lambda}}$ *and* $\sqrt{-i\lambda}\tan\sqrt{-i\lambda}$ *are to have determinations which are positive when* λ *is purely imaginary and contained in the interval* $(0, i\pi^2/4)$.

LEMMA 2. *Let* $f(\lambda)$ *and* $h(\lambda)$ *belong to* $L(-\infty, \infty)$ *and denote by* $F(u)$ *and* $H(u)$ *their Fourier transforms:*

$$F(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\lambda)e^{-i\lambda u}\, d\lambda, \qquad H(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(\lambda)e^{-i\lambda u}\, d\lambda. \tag{1.9}$$

If $H(u)$ *belongs to* $L(-\infty, \infty)$ *then Parseval's relation holds; namely,*

$$\int_{-\infty}^{\infty} f(\lambda)h(\lambda)\, d\lambda = \int_{-\infty}^{\infty} F(u)H(-u)\, du. \tag{1.10}$$

LEMMA 3. *Let $q(x)$ be a real-valued Wiener measurable functional of $x(\cdot)$ and assume that the function*

$$h(\lambda) = \int_C e^{-i\lambda q(x)}\, d_w x, \tag{1.11}$$

as well as its Fourier transform $H(u)$, belong to $L_1(-\infty, \infty)$. Let $F(u)$ be any (real or complex) measurable function defined on $-\infty < u < \infty$. Then a necessary and sufficient condition that

$$F[q(x)]$$

be Wiener summable over C is that

$$F(u)H(-u) \in L_1(-\infty, \infty). \tag{1.12}$$

Moreover, if this condition is satisfied,

$$\int_C F[q(x)]\, d_w x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(u)H(-u)\, du. \tag{1.13}$$

LEMMA 4. *For each real value of ξ the function $h(\lambda, \xi)$ in (1.5) belongs to L_1 $(-\infty < \lambda < \infty)$, and its Fourier transform $H(u, \xi)$ belongs to $L_1(-\infty < u < \infty)$.*

LEMMA 5. *The Fourier transform $H(u, \xi)$ of the function $h(\lambda, \xi)$ of (1.5) vanishes for $u > 0$.*

Lemma 1 will be proved in the next section. As mentioned earlier, Lemma 2 is known, [4]. Lemma 3 will be proved in Section 3; its proof will use Lemma 2 and a familiar type of smoothing argument. The proofs of Lemmas 4 and 5 will be given in Sections 4 and 5 respectively. On applying Lemmas 4 and 5, together with Lemma 3 for $q(x) = \int_0^1 \{x(t) + \xi\}^2 dt$, we obtain Theorem 2. In Section 6 we specialize the function F of Theorem 2 and integrate over the product space $[x(\cdot), \xi]$ to obtain a measure for sets in the product space for which $\int_0^1 \{x(t) + \xi\}^2 dt < R^2$. In Section 7 we evaluate the Fourier transform of $h(\lambda, 0)$ in terms of the theta-function and derive Theorem 1 from Theorem 2. In the final section we derive Corollary 1 as well as the following corollary to Theorem 1:

COROLLARY 2. *Relation (1.3) of Theorem 1 may be written in the following alternative form*

$$\int_C F\left[\int_0^1 x^2(t)\, dt\right] d_w x = \frac{1}{\sqrt{2\pi}} \int_0^\infty F\left(\frac{1}{s^2}\right) \sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} (4n+1) e^{-(n+\frac{1}{2})^2 s^2}\, ds. \tag{1.14}$$

2. Proof of Lemma 1. In an earlier paper [5] on the evaluation of Wiener integrals, the authors have obtained the following result:

RESULT 1. *Let $g(t)$ be a real-valued Lebesgue measurable function belonging to L_2 on $0 \le t \le 1$, and let μ be real and on the interval*

$$-\frac{\pi}{2} < \mu < \frac{\pi}{2}. \tag{2.1}$$

Then

$$\int_C e^{\mu^2 \int_0^1 [x(t)+g(t)]^2\,dt}\,d_w x = \sqrt{\sec \mu}\; e^{\mu^4 \int_0^1 [G(t,u)\sec \mu(t-1)]^2\,dt\; \mu^2 \int_0^1 g^2(t)\,dt} \tag{2.2}$$

where $G(t, \mu)$ *is defined by*

$$G(t, \mu) = \int_1^t g(s) \cos \mu(s-1)\,ds. \tag{2.3}$$

On choosing $g(t) = \xi$, where ξ is a real constant, we obtain

$$\begin{aligned} &\mu^4 \int_0^1 \left[\sec \mu(t-1) \int_1^t \xi \cos \mu(s-1)\,ds\right]^2 dt + \mu^2 \xi^2 \\ &= \mu^2 \xi^2 \int_0^1 \sec^2 \mu(t-1) \sin^2 \mu(t-1)\,dt + \mu^2 \xi^2 \qquad (2.4) \\ &= \mu^2 \xi^2 \int_0^1 [\tan^2 \mu(t-1) + 1]\,dt = \mu \xi^2 \tan \mu, \end{aligned}$$

and hence we have

RESULT 2. *Let* ξ *be an arbitrary real number. Then*

$$\int_C e^{\mu^2 \int_0^1 [x(t)+\xi]^2\,dt}\,d_w x = \sqrt{\sec \mu}\; e^{\xi^2 \mu \tan \mu}, \qquad -\frac{\pi}{2} < \mu < \frac{\pi}{2}. \tag{2.5}$$

It is now a simple matter to obtain Lemma 1. We consider (2.5) for μ complex and show easily by analytic continuation that the relation (2.5) persists when $\mu^2 = -i\lambda$ with λ real, $-\infty < \lambda < \infty$. The argument follows almost word for word that given in the final paragraph of [3]. With $\mu^2 = -i\lambda$, (2.5) yields Lemma 1.

3. Proof of Lemma 3. First assume that $F(u)$ is the Fourier transform of a function $f(\lambda)$ belonging to $L_1(-\infty < \lambda < \infty)$,

$$F(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\lambda) e^{-iu\lambda}\,d\lambda.$$

Then

$$\begin{aligned} \int_C F[q(x)]\,d_w x &= \frac{1}{\sqrt{2\pi}} \int_C \left[\int_{-\infty}^{\infty} f(\lambda) e^{-iuq(x)}\,d\lambda\right] d_w x \\ \text{(by Fubini's theorem)} \qquad &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\lambda) \left[\int_C e^{-iuq(x)}\,d_w x\right] d\lambda \\ \text{(by (1.11))} \qquad &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\lambda) h(\lambda)\,d\lambda \\ \text{(by Lemma 2)} \qquad &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(u) H(-u)\,du. \end{aligned}$$

Thus the result (1.13) of Lemma 3 holds for every function $F(u)$ whose Fourier transform belongs to $L_1(-\infty, \infty)$. By use of a familiar argument we shall carry through the remainder of the proof of Lemma 3 in successive stages.

STEP 1. We establish (1.13) when $F(u)$ is a real bounded measurable function vanishing outside a finite interval. For $\delta > 0$ we form the function

$$F_\delta(u) = \frac{1}{\delta}\int_u^{u+\delta} F(v)\,dv.$$

Then $F_\delta(u)$ is absolutely continuous, it vanishes outside a finite interval, and it has an essentially bounded derivative. Hence the Fourier transform of $F_\delta(u)$ belongs to $L(-\infty, \infty)$, and hence (1.13) holds for $F_\delta(u)$. By bounded convergence it holds for $F(u)$.

STEP 2. We next establish Lemma 3 for real non-negative $F(u)$. This is established in the customary way by use of the principle of monotone convergence. The existence of one side of (1.13) proves the existence of the other, and hence, we obtain the desired necessary and sufficient condition of Wiener summability.

STEP 3. The case of general complex-valued functions is established by combining the four parts, real positive, real negative, pure imaginary positive, and pure imaginary negative.

This establishes Lemma 3 in its general form, including summability of the integral.

4. Proof of Lemma 4.

In order to show that the Fourier transform of (1.5) belongs to $L(-\infty, 0)$ we apply a theorem of Paley and Wiener [6, Theorem IV, p. 7] which states that:

Let $p(s) = p(\sigma + i\lambda)$ be analytic over $\mu_1 \leq \sigma \leq \mu_2$, and let

$$\int_{-\infty}^{\infty} |\, p(\sigma + i\lambda)\,|^2\, d\lambda < \text{const.} \qquad [\mu_1 \leq \sigma \leq \mu_2]. \tag{4.1}$$

Then there exists a measurable function $P(u)$, such that

$$\int_{-\infty}^{\infty} |\, P(u)\,|^2 e^{2\mu_j u}\, du < \infty \qquad j = 1, 2, \tag{4.2}$$

and that over the closed interval $\mu_1 \leq \sigma \leq \mu_2$,

$$p(\sigma + i\lambda) = \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}}\int_{-A}^{A} P(u)e^{u(\sigma+i\lambda)}\,du. \tag{4.3}$$

We take (for fixed real ξ)

$$p(\sigma + i\lambda) = \frac{e^{\xi^2\sqrt{\sigma+i\lambda}\,\tan\sqrt{\sigma+i\lambda}}}{\sqrt{\cos\sqrt{\sigma + i\lambda}}} \tag{4.4}$$

and we note that p is analytic in the half plane $\sigma < \pi^2/4$. (We choose the branches of the square roots which are positive for $0 \leq \sigma < \pi^2/4, \lambda = 0$). Moreover, if

$$0 \leq \sigma < \frac{\pi^2}{4} \tag{4.5}$$

we have

$$(4.6)\qquad I_\sigma \equiv \int_{-\infty}^{\infty} |\, p(\sigma + i\lambda)\,|^2\, d\lambda$$

$$= \int_{-\infty}^{\infty} \frac{e^{2\xi^2 \operatorname{Re}(\sqrt{\sigma+i\lambda}\,\tan\sqrt{\sigma+i\lambda})}}{|\cos\sqrt{\sigma+i\lambda}\,|}\, d\lambda$$

$$= \int_{-\infty}^{\infty} \frac{e^{2\xi^2 \operatorname{Re}(\sqrt{\sigma+i\lambda}\,\tan\sqrt{\sigma+i\lambda})}}{[\cosh^2(\operatorname{Im}\sqrt{\sigma+i\lambda}) - \sin^2(\operatorname{Re}\sqrt{\sigma+i\lambda})]^{1/2}}\, d\lambda$$

where we have used the simple relation

$$\begin{aligned} |\cos(x+iy)| &= |\cos x \cosh y - i \sin x \sinh y| \\ &= [\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y]^{\frac{1}{2}} \\ &= [\cos^2 x \cosh^2 y + \sin^2 x\,(\cosh^2 y - 1)]^{\frac{1}{2}} \\ &= [\cosh^2 y - \sin^2 x]^{\frac{1}{2}}. \end{aligned}$$

Denoting by u and v the real and imaginary parts of $\sqrt{\sigma + i\lambda}$,

$$(4.7)\qquad u = \operatorname{Re}\sqrt{\sigma + i\lambda}, \qquad v = \operatorname{Im}\sqrt{\sigma + i\lambda}$$

we have

$$(4.8)\quad \operatorname{Re}(\sqrt{\sigma + i\lambda}\,\tan\sqrt{\sigma + i\lambda}) = \operatorname{Re}\{(u+iv)\tan(u+iv)\}$$

$$= \operatorname{Re}\left\{(u+iv)\,\frac{\tan u + i\tanh v}{1 - i\tan u\tanh v}\right\}$$

$$= \frac{u\tan u(1 - \tanh^2 v) - v\tanh v\,(1 + \tan^2 u)}{1 + \tan^2 u\tanh^2 v}$$

$$= \frac{u\tan u\operatorname{sech}^2 v - v\tanh v\sec^2 u}{1 + \tan^2 u\tanh^2 v}$$

$$= \frac{u\sin u\cos u - v\sinh v\cosh v}{\cos^2 u\cosh^2 v + \sin^2 u\sinh^2 v}.$$

By (4.7) we also have

$$u^2 - v^2 + 2iuv = \sigma + i\lambda, \quad \text{or} \quad \sigma = u^2 - v^2,\ \lambda = 2uv$$

so that

$$u = \sqrt{v^2 + \sigma}, \qquad \lambda = 2v\sqrt{v^2 + \sigma}.$$

Hence if we apply the transformation

$$(4.9)\qquad \lambda = 2v\sqrt{v^2 + \sigma}, \qquad d\lambda = \frac{2(2v^2 + \sigma)\,dv}{\sqrt{v^2 + \sigma}}$$

to the final integral in (4.6), with σ as before in (4.5), we obtain

$$(4.10)\qquad I_\sigma = \int_{-\infty}^{\infty} \frac{2(2v^2 + \sigma)\exp\{2\xi^2 A(v,\sigma)\}}{\sqrt{v^2 + \sigma}\,[\cosh^2 v - \sin^2\sqrt{v^2 + \sigma}]^{\frac{1}{2}}}\, dv$$

where

$$A(v,\sigma) = \frac{\sqrt{v^2+\sigma}\sin\sqrt{v^2+\sigma}\cos\sqrt{v^2+\sigma} - v\sinh v\cosh v}{\cos^2\sqrt{v^2+\sigma}\cosh^2 v + \sin^2\sqrt{v^2+\sigma}\sinh^2 v} \tag{4.11}$$

We shall show that $A(v,\sigma)$ is bounded above in

$$0 \le \sigma \le \sigma_0 < \frac{\pi^2}{4}, \qquad 0 \le v < \infty. \tag{4.12}$$

(Since it is an even function of v, this will mean that it is bounded above in $0 \le \sigma \le \sigma_0$, $-\infty < v < \infty$). First we look at the numerator

$$\sqrt{v^2+\sigma}\sin\sqrt{v^2+\sigma}\cos\sqrt{v^2+\sigma} - v\sinh v\cosh v$$
$$\le \sqrt{v^2+\sigma} - v\frac{e^{2v}-e^{-2v}}{4} \le v + \frac{\pi}{2} - v\frac{e^{2v}-e^{-2v}}{4}$$

for (v,σ) in (4.12). Since

$$v + \frac{\pi}{2} - v\frac{e^{2v}-e^{-2v}}{4} \to -\infty \quad \text{as} \quad v \to \infty$$

and since it is continuous and independent of σ, it follows that there exists a positive constant M_0 such that the numerator of $A(v,\sigma)$ is less than M_0,

$$\sqrt{v^2+\sigma}\sin\sqrt{v^2+\sigma}\cos\sqrt{v^2+\sigma} - v\sinh v\cosh v \le M_0 \tag{4.13}$$

for (v,σ) in (4.12). Next we look at the denominator of $A(v,\sigma)$. It is easily seen that the denominator does not vanish in

$$0 \le \sigma \le \sigma_0 < \frac{\pi^2}{4}, \qquad 0 \le v \le 1, \tag{4.14}$$

and since it is continuous there it has a positive lower bound there. Also for $0 \le \sigma \le \sigma_0$, $1 \le v < \infty$ we have

$$\cosh v \ge 1, \qquad \sinh v \ge 1$$

and hence

$$\cos^2\sqrt{v^2+\sigma}\cosh^2 v + \sin^2\sqrt{v^2+\sigma}\sinh^2 v$$
$$\ge \cos^2\sqrt{v^2+\sigma} + \sin^2\sqrt{v^2+\sigma} = 1$$

for $0 \le \sigma \le \sigma_0$, $1 \le v < \infty$. Hence there exists a positive constant ϵ_0 such that the denominator of $A(v,\sigma)$ is greater than ϵ_0 throughout (4.12). Hence

$$A(v,\sigma) \le \frac{M_0}{\epsilon_0} = M \tag{4.15}$$

for (v,σ) in (4.12) and hence by (4.10)

$$I_\sigma \le e^{2t^2M}\int_{-\infty}^{\infty}\frac{2(2v^2+\sigma)\,dv}{\sqrt{v^2+\sigma}\,[\cosh^2 v - \sin^2\sqrt{v^2+\sigma}]^{\frac{1}{2}}} \tag{4.16}$$
$$= 4e^{2t^2M}\int_0^{\infty}\frac{2v^2+\sigma}{\sqrt{v^2+\sigma}\,[\cosh^2 v - \sin^2\sqrt{v^2+\sigma}]^{\frac{1}{2}}}\,dv$$

for

$$0 \leq \sigma \leq \sigma_0 < \frac{\pi^2}{4}. \tag{4.17}$$

Now since σ is non-negative

$$\frac{2v^2+\sigma}{\sqrt{v^2+\sigma}} \leq \frac{2v^2+2\sigma}{\sqrt{v^2+\sigma}} = 2\sqrt{v^2+\sigma}$$

and hence

$$I_\sigma \leq 8e^{2t^2M} \int_0^\infty \frac{\sqrt{v^2+\sigma}}{[\cosh^2 v - \sin^2\sqrt{v^2+\sigma}]^{\frac{1}{2}}}\, dv \tag{4.18}$$

$$\leq 8e^{2t^2M} \int_0^1 \frac{\sqrt{v^2+\sigma}\, dv}{[\cosh^2 v - \sin^2\sqrt{v^2+\sigma}]^{\frac{1}{2}}} + 8e^{2t^2M} \int_1^\infty \frac{\sqrt{v^2+\sigma_0}\, dv}{[\cosh^2 v - 1]^{\frac{1}{2}}}.$$

Now $\cosh^2 v - \sin^2\sqrt{v^2+\sigma}$ is continuous and non-vanishing in the rectangle $0 \leq v \leq 1, 0 \leq \sigma \leq \sigma_0$, since it can vanish only when both terms equal unity, and this does not occur simultaneously in the rectangle. Thus the first integrand is bounded in v, σ in the rectangle and the first integral is bounded in σ in (4.17). Moreover the second integral converges since its denominator is sinh v which behaves like e^v at ∞, and this term is independent of σ and hence is bounded in σ. Hence the hypotheses of the theorem of Paley and Wiener quoted at the beginning of this section are satisfied with $\mu_1 = 0$, $\mu_2 = \sigma_0$, and hence there exists a measurable function $P(u)$ such that

$$\int_{-\infty}^{\infty} |P(u)|^2\, du < \infty, \qquad \int_{-\infty}^{\infty} |P(u)|^2 e^{2\sigma_0 u}\, du < \infty, \tag{4.19}$$

and (4.3) holds on (4.17). Also by the second of the relations in (4.19) and Schwarz's inequality we have

$$\int_0^\infty |P(u)|\, du \leq \left[\int_0^\infty |P(u)|^2 e^{2\sigma_0 u}\, du \cdot \int_0^\infty e^{-2\sigma_0 u}\, du\right]^{\frac{1}{2}} < \infty \tag{4.20}$$

so that $P(u) \in L(0, \infty)$.

For the proof of Lemma 4 it remains to show that (1.5) belongs to $L(-\infty < \lambda < \infty)$ and to identify $P(u)$ with $H(-u, \xi)$. To show that (1.5) belongs to $L(-\infty, \infty)$ we find as in (4.6), (4.10), and (4.18) for $\sigma = 0$:

$$\int_{-\infty}^{\infty} \frac{|e^{t^2\sqrt{i\lambda}\tan\sqrt{i\lambda}}|}{|\cos\sqrt{i\lambda}|^{\frac{1}{2}}}\, d\lambda \leq 8e^{t^2M} \int_0^1 \frac{v\, dv}{[\cosh^2 v - \sin^2 v]^{\frac{1}{2}}} + 8e^{t^2M} \int_1^\infty \frac{v\, dv}{[\cosh^2 v - 1]^{\frac{1}{2}}}.$$

Hence (1.5) belongs to $L(-\infty, \infty)$. Next applying (4.3) for $\sigma = 0$ and

$$p(i\lambda) = \frac{e^{t^2\sqrt{i\lambda}\tan\sqrt{i\lambda}}}{\sqrt{\cos\sqrt{i\lambda}}}$$

we see that

$$\frac{e^{\xi^2\sqrt{-i\lambda}\tan\sqrt{-i\lambda}}}{\sqrt{\cos\sqrt{-i\lambda}}} = h(\lambda;\xi) = \text{l.i.m.}_{A\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A} P(u)e^{-i\lambda u}\,du \tag{4.21}$$

for $-\infty < \lambda < \infty$. Thus $P(-u)$ is the ordinary L_2 Fourier transform of $h(\lambda;\xi)$ and hence

$$P(-u) = H(\mu,\xi) \tag{4.22}$$

almost everywhere in u. Since we have shown (in 4.20)) that $P(u) \in L(0,\infty)$ it follows that $H(u,\xi) \in L(-\infty < u < 0)$. In Lemma 5 we show $H(u,\xi) = 0$ if $u > 0$. This concludes the proof of Lemma 4.

5. Proof of Lemma 5.

We have shown that the function

$$\sqrt{\sec\sqrt{-i\lambda}}\,e^{\xi^2\sqrt{-i\lambda}\tan\sqrt{-i\lambda}}$$

belongs to $L(-\infty,\infty)$ for every real value of ξ. In this section we want to show that its Fourier transform

$$H(u,\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{\xi^2\sqrt{-i\lambda}\tan\sqrt{-i\lambda}}e^{-i\lambda u}}{\sqrt{\cos\sqrt{-i\lambda}}}\,d\lambda \tag{5.1}$$

is 0 for $u > 0$. For this purpose let $z^2 = -i\lambda$, and let C_1 be a contour coming from ∞ in the first quadrant along the line $\arg z = \pi/4$ to the origin, and then out along the line $\arg z = -\pi/4$ to ∞ in the fourth quadrant. Then clearly

$$H(-u,\xi) = \frac{i}{\sqrt{2\pi}}\int_{C_1}\frac{e^{-uz^2}2ze^{\xi^2 z\tan z}}{\sqrt{\cos z}}\,dz. \tag{5.2}$$

Now from (5.2) we see at once that

$$H(-u,\xi) = 0 \qquad \text{if } u < 0, \tag{5.3}$$

for in that case we can deform the contour into the imaginary axis, and since the integrand is odd, the upper half is the negative of the lower half, and they cancel.

This yields Lemma 5. Theorem 2 follows at once from Lemmas 1, 3, 4, and 5, with $q(x) = \int_0^1 \{x(t)+\xi\}^2 dt$ in Lemma 3.

6. A measure for $\int_0^1 \{x(t)+\xi\}^2\,dt < R^2$ in the product-space $[x(\cdot),\xi]$.

Theorem 2 enables us to obtain very easily a measure in the product space $[x(\cdot),\xi]$. Let R be a positive number. First consider ξ a fixed real number and denote by $S(R,\xi)$ the set of functions $x(\cdot)$ of C for which

$$\int_0^1 [x(t)+\xi]^2\,dt < R^2. \tag{6.1}$$

Then by Theorem 2 with $F(u) = 1$ for $0 < u < R^2$ and zero elsewhere we have

$$\text{meas}_w S(R, \xi) = \int_0^{R^2} H(-u, \xi)\, du \tag{6.2}$$

where $H(-u, \xi)$ is defined by (1.6).

We now let ξ vary with a Gaussian distribution of variance unity and we consider the product space $[x(\cdot), \xi]$. In this product space we consider the set $\sum(R)$ for which (6.1) holds. We define the measure of $\sum(R)$ to be the double integral

$$\frac{1}{\sqrt{\pi}} \int\int^{w} e^{-\xi^2}\, d_w x \cdot d\xi \tag{6.3}$$

taken over the portion of the product space $[x(\cdot), \xi]$ for which (6.1) holds. By treating (6.3) as a repeated integral we can write it as

$$\text{meas} \sum (R) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\xi^2}\, d\xi \int_0^{R^2} H(-u, \xi)\, du \tag{6.4}$$

with $H(-u, \xi)$ defined in (1.6). This gives the measure of $\sum(R)$ in terms of ordinary Lebesgue integrals.

7. Proof of Theorem 1.

With $\xi = 0$, relation (1.7) of Theorem 2 becomes

$$\int_c^{w} F\left[\int_0^1 x^2(t)\, dt\right] d_w x = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} F(u) H(-u)\, du \tag{7.1}$$

where

$$H(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\sec \sqrt{-i\lambda}}\; e^{-i\lambda u}\, d\lambda. \tag{7.2}$$

By (5.2) $H(u)$ can also be written in the form

$$H(-u) = \frac{i}{\sqrt{2\pi}} \int_{C_1} \frac{2ze^{-uz^2}}{\sqrt{\cos z}}\, dz \tag{7.3}$$

where C_1 is a contour coming from ∞ in the first quadrant along the line $\arg z = \pi/4$ to the origin, and then going out along the line $\arg z = -\pi/4$ to ∞ in the fourth quadrant. In order to express $H(-u)$ for $u > 0$ in terms of the theta function we deform the contour C_1 into the contour defined as follows: We come in from ∞ along the positive real axis to the origin, by-passing each of the points $\pi/2 + n\pi$, $n = 0, 1, 2, \cdots$ by going in a counterclockwise direction along the upper half of a circle of radius ϵ with center at $z = \pi/2 + n\pi$. After reaching the origin we return to ∞ along the positive real axis again by-passing the points $\pi/2 + n\pi$, this time by going along the lower halves of the same circles. Now $\sqrt{\cos z}$ will be real and have the same value on the upper and lower branches on the real axis for

$$0 \leq x < \frac{\pi}{2}, \qquad \frac{3\pi}{2} < x < \frac{5\pi}{2}, \qquad \frac{7\pi}{2} < x < \frac{9\pi}{2}, \cdots$$

but will be purely imaginary and have opposite signs on the upper and lower branches when

$$\frac{\pi}{2} < x < \frac{3\pi}{2}, \qquad \frac{5\pi}{2} < x < \frac{7\pi}{2}, \cdots.$$

Hence the upper and lower contours cancel on half the intervals but double up on the other half and hence we have on taking the limit as $\epsilon \to 0^+$,

$$H(-u) = \frac{4}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \int_{\pi/2+2n\pi}^{3\pi/2+2n\pi} (-1)^n \frac{xe^{-ux^2}}{\sqrt{-\cos x}}\, dx, \qquad u > 0. \tag{7.4}$$

Putting $t = x - (2n+1)\pi$, we have

$$\begin{aligned} H(-u) &= \frac{4}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \int_{\pi/2}^{\pi/2} (-1)^n \frac{[t + (2n+1)\pi]}{\sqrt{\cos t}} \\ &\qquad \cdot \exp\{-u[t + (2n+1)\pi]^2\}\, dt \\ &= \frac{4}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \int_{0}^{\pi/2} (-1)^n \frac{t + (2n+1)\pi}{\sqrt{\cos t}} \\ &\qquad \cdot \exp\{-u[t + (2n+1)\pi]^2\}\, dt, \qquad u > 0. \end{aligned} \tag{7.5}$$

Now consider the sum

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} (-1)^n \exp\{-u[t + (2n+1)\pi]^2\} \\ &\qquad = e^{-ut^2} \sum_{n=-\infty}^{\infty} (-1)^n e^{-\pi^2 u(2n+1)^2} e^{-(2n+1)2u\pi t}. \end{aligned} \tag{7.6}$$

Let

$$q = e^{-4\pi^2 u} \tag{7.7}$$

and

$$z = 2\pi i u t \tag{7.8}$$

Then (7.6) becomes

$$e^{-ut^2} \sum_{n=-\infty}^{\infty} (-1)^n q^{(2n+1)^2/4} e^{i(2n+1)z}. \tag{7.9}$$

Now the standard definition of the theta function of the first kind is

$$\begin{aligned} \vartheta_1(z \mid \tau) = \vartheta_1(z, q) &= 2 \sum_{n=0}^{\infty} (-1)^n q^{(2n+1)^2/4} \sin(2n+1)z \\ &= \frac{1}{i} \sum_{n=-\infty}^{\infty} (-1)^n q^{(2n+1)^2/4} e^{i(2n+1)z} \end{aligned} \tag{7.10}$$

where

$$q = e^{\pi i \tau} \text{ and } Im\ \tau > 0. \tag{7.11}$$

Hence (7.6) is equal to

$$ie^{-ut^2}\vartheta_1(2\pi iut, e^{-4\pi^2 u}). \tag{7.12}$$

On differentiating (7.6) and (7.12) with respect to t and dividing by $-2u$ we find

$$\begin{aligned}\sum_{n=-\infty}^{\infty} (-1)^n[t + (2n+1)\pi]\cdot e^{-u[t+(2n+1)\pi]^2} \\ = -\frac{i}{2u}\frac{\partial}{\partial t}\{e^{-ut^2}\vartheta_1(2\pi iut, e^{-4\pi^2 u})\} \\ = -\frac{i}{2u}\frac{\partial}{\partial t}\{e^{-ut^2}\vartheta_1(2\pi iut \mid 4\pi iu)\},\end{aligned} \tag{7.13}$$

the term-by-term differentiation being easily justified for $u > 0$. On the basis of Jacobi's imaginary transformation the theta function is known to satisfy the relation

$$\vartheta_1(z \mid \tau) = -\frac{i}{\sqrt{-i\tau}} e^{z^2/\pi i\tau}\vartheta_1\left(-\frac{z}{\tau}\middle| -\frac{1}{\tau}\right), \tag{7.14}$$

where $\sqrt{-i\tau}$ is positive for Re $\tau = 0$, $Im\tau > 0$; see for example [7, p. 475]. Using (7.14) with

$$z = 2\pi iut \quad \text{and} \quad \tau = 4\pi iu, \qquad (u > 0)$$

we have

$$\vartheta_1(2\pi iut \mid 4\pi iu) = -\frac{i}{\sqrt{4\pi u}} e^{ut^2}\vartheta_1\left(-\frac{t}{2}\middle| \frac{i}{4\pi u}\right) \tag{7.15}$$

and hence (7.13) yields

$$\begin{aligned}\sum_{n=-\infty}^{\infty} (-1)^n[t + (2n+1)\pi]e^{-u[t+(2n+1)\pi]^2} \\ = -\frac{1}{4\sqrt{\pi u^3}}\frac{\partial}{\partial t}\vartheta_1\left(-\frac{t}{2}\middle| \frac{i}{4\pi u}\right) \\ = +\frac{1}{8\sqrt{\pi u^3}}\vartheta_1'\left(\frac{t}{2}\middle| \frac{i}{4\pi u}\right) \\ = \frac{1}{8\sqrt{\pi u^3}}\vartheta_1'\left(\frac{t}{2}, e^{-1/4u}\right)\end{aligned} \tag{7.16}$$

where we have used among other things the fact that $\vartheta_1(z \mid \tau)$ is an odd function of z.

Inserting (7.16) into (7.5) and performing a justifiable interchange of order of summation and integration we find that $H(-u)$ has the form

$$H(-u) = \frac{1}{2^3\pi u^3}\int_0^{\pi/2} \frac{\vartheta_1'\left(\frac{t}{2}, e^{-1/4u}\right)}{\sqrt{\cos t}}\, dt, \quad u > 0. \tag{7.17}$$

This yields Theorem 1.

8. Derivation of Corollaries 1 and 2.

We obtain relation (0.2) of Corollary 1 by inserting in equation (1.3) the function $F(u) = 1$ for $0 < u < R^2$ and $= 0$ elsewhere. To obtain Corollary 2 and the asymptotic development (0.5) of Corollary 1 we proceed as follows.

Using (7.17) and the series development (7.10) for the theta function we see that

$$H(-u) = \frac{1}{2^{\frac{1}{2}}\pi u^{\frac{1}{2}}} \int_0^{\pi/2} \frac{\vartheta_1'\left(\frac{t}{2}, e^{-1/4u}\right)}{\sqrt{\cos t}}\, dt$$

(8.1)

$$= \frac{2}{2^{\frac{1}{2}}\pi u^{\frac{1}{2}}} \int_0^{\pi/2} \frac{\sum_{n=0}^{\infty} (-1)^n (2n+1) \cos (2n+1)\frac{t}{2} \cdot e^{-(2n+1)^2/16u}}{\sqrt{\cos t}}\, dt,$$

for $u > 0$. Now Mehler's simplified form of Dirichlet's integral for Legendre functions states that

$$P_n(\cos \alpha) = \frac{2}{\pi} \int_0^{\alpha} \frac{\cos (n + \frac{1}{2})t}{\{2(\cos t - \cos \alpha)\}^{\frac{1}{2}}}\, dt \tag{8.2}$$

where $P_n(z)$ is the Legendre function of degree n, see [7, p. 315]. For $\alpha = \frac{\pi}{2}$ this yields

$$\int_0^{\pi/2} \frac{\cos (n + \frac{1}{2})t}{\sqrt{\cos t}}\, dt = \frac{\pi}{\sqrt{2}} P_n(0). \tag{8.3}$$

Also it is known that

$$P_n(0) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \text{ is an odd positive integer} \\ (-1)^{n/2} \dfrac{1 \cdot 3 \cdot \cdots (n-1)}{2 \cdot 4 \cdots n} & \text{if } n \text{ is an even positive integer,} \end{cases} \tag{8.4}$$

see, for example [7, p. 302]. Using (8.3) and (8.4) and integrating the series in (8.1) term by term we find

$$H(-u) = \frac{1}{2u^{\frac{1}{2}}} \sum_{\nu=0}^{\infty} (-1)^\nu e^{-(4\nu+1)^2/16u} (4\nu + 1) \frac{1 \cdot 3 \cdots (2\nu - 1)}{2 \cdot 4 \cdots (2\nu)} \tag{8.5}$$

where for $\nu = 0$ we understand the quotient

$$\frac{1 \cdot 3 \cdots (2\nu - 1)}{2 \cdot 4 \cdots (2\nu)} \tag{8.6}$$

to be unity. Hence returning to relation (1.3) we find

$$\int_c^{w} F\left[\int_0^1 x^2(t)\, dt\right] d_w x = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} F(u) H(-u)\, du$$

$$(8.7) \qquad = \frac{1}{2^{\frac{1}{2}}\pi^{\frac{1}{2}}} \int_0^\infty \frac{F(u)}{u^{\frac{1}{2}}} \sum_{\nu=0}^\infty \frac{(-1)^\nu 1\cdot 3\cdots(2\nu-1)}{2\cdot 4\cdots(2\nu)} (4\nu+1) e^{-(4\nu+1)/16u}\, du$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty F\left(\frac{1}{s^2}\right) \sum_{\nu=0}^\infty \frac{(-1)^\nu 1\cdot 3\cdots(2\nu-1)}{2\cdot 4\cdots(2\nu)} (4\nu+1) e^{-(\nu+\frac{1}{4})^2/s^2}\, ds.$$

This yields Corollary 2.

To obtain the asymptotic development (0.2) of Corollary 1 we again choose $F(u) = 1$ for $0 < u < R^2$ and $= 0$ elsewhere. Then after an interchange of order of integration and summation (8.7) becomes

$$(8.8) \quad \text{meas}_w S_R = \frac{1}{\sqrt{2\pi}} \sum_{\nu=0}^\infty (-1)^\nu \frac{1\cdot 3\cdots(2\nu-1)}{2\cdot 4\cdots(2\nu)} (4\nu+1) \cdot \int_{1/R}^\infty e^{-(\nu+\frac{1}{4})^2/s^2}\, ds$$

$$= \frac{4}{\sqrt{2\pi}} \sum_{\nu=0}^\infty (-1)^\nu \frac{1\cdot 3\cdots(2\nu-1)}{2\cdot 4\cdots(2\nu)} \operatorname{Erfc}\left(\frac{\nu+\frac{1}{4}}{R}\right),$$

where $Erfc(u)$ is the well-known error function of the theory of probability; namely,

$$(8.9) \qquad Erfc(u) = \int_u^\infty e^{-v^2}\, dv \qquad (u \text{ real}),$$

Now it is known [7, p. 152] that

$$(8.10) \qquad Erfc(u) \sim \tfrac{1}{2} e^{-u^2} \frac{1}{u} \left\{1 + \sum_{n=1}^\infty (-1)^n \frac{1\cdot 3\cdots(2n-1)}{2^n u^{2n}}\right\}$$

for large positive values of u. Inserting this into (8.8) and noting that later terms are asymptotic to zero in comparison with the first term, we find that $\text{meas}_w S_R$ has the asymptotic development given in (0.5).

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TABLE OF COEFFICENTS FOR DIFFERENCES IN TERMS OF THE DERIVATIVES

By Herbert E. Salzer[1]

The following table lists the exact values of the coefficients $B_{m,s}$ for $m = 1, 2, \cdots 20$ and $s = m, \cdots 20$, in the formula of Markoff expressing the mth advancing difference in terms of the derivatives according to the equation

$$\Delta_h^m f(a) = \sum_{s=m}^{n-1} B_{m,s} h^s D^s f(a) + B_{m,n} h^n D^n f(\eta)$$

(Here h represents the tabular interval.) Expressed in the notation of Milne-Thomson, $B_{m,s} = B_{s-m}^{(-m)}/(s-m)!$ where $B_{s-m}^{(-m)}$ is the $(s-m)$th Bernoulli number of order $-m$. In the notation of Jordan, $B_{m,s} = m!\,\mathfrak{S}_s^m/s!$ where $\mathfrak{S}_s^m$ denotes the "Stirling number of the second kind" or $[\Delta^m x^s/m!]_{x=0}$. Jordan's definition is more convenient from the standpoint of computation.

The coefficients $B_{m,s}$ were calculated as follows. The Stirling numbers of the second kind $\mathfrak{S}_s^m$ were obtained from the recursion formula $\mathfrak{S}_{s+1}^m = \mathfrak{S}_s^{m-1} + m\mathfrak{S}_s^m$ beginning with $\mathfrak{S}_s^1 = 1$ and $\mathfrak{S}_s^m = 0$ for $m > s$, up to $s = 20$. (The numbers $\mathfrak{S}_s^m$ are given correctly up to $s = 12$ in Jordan, p. 170 and in H. T. Davis, "Tables of the Higher Mathematical Functions," vol. II, p. 212.) Then all the quantities $\mathfrak{S}_s^m$ were checked by the formula in Jordan, p. 188, equation (15) which reads (with a slight alteration in Jordan's unusual notation for summation):

$$\sum_{m=1}^{s+1} \mathfrak{S}_{s+1}^m = \sum_{m=1}^{s} (1+m)\mathfrak{S}_s^m$$

Then $B_{m,s}$ was obtained from $\mathfrak{S}_s^m$ through multiplication by $m!/s!$ The multiplication was checked by division. All values of $B_{m,s}$ are believed to be in lowest terms.

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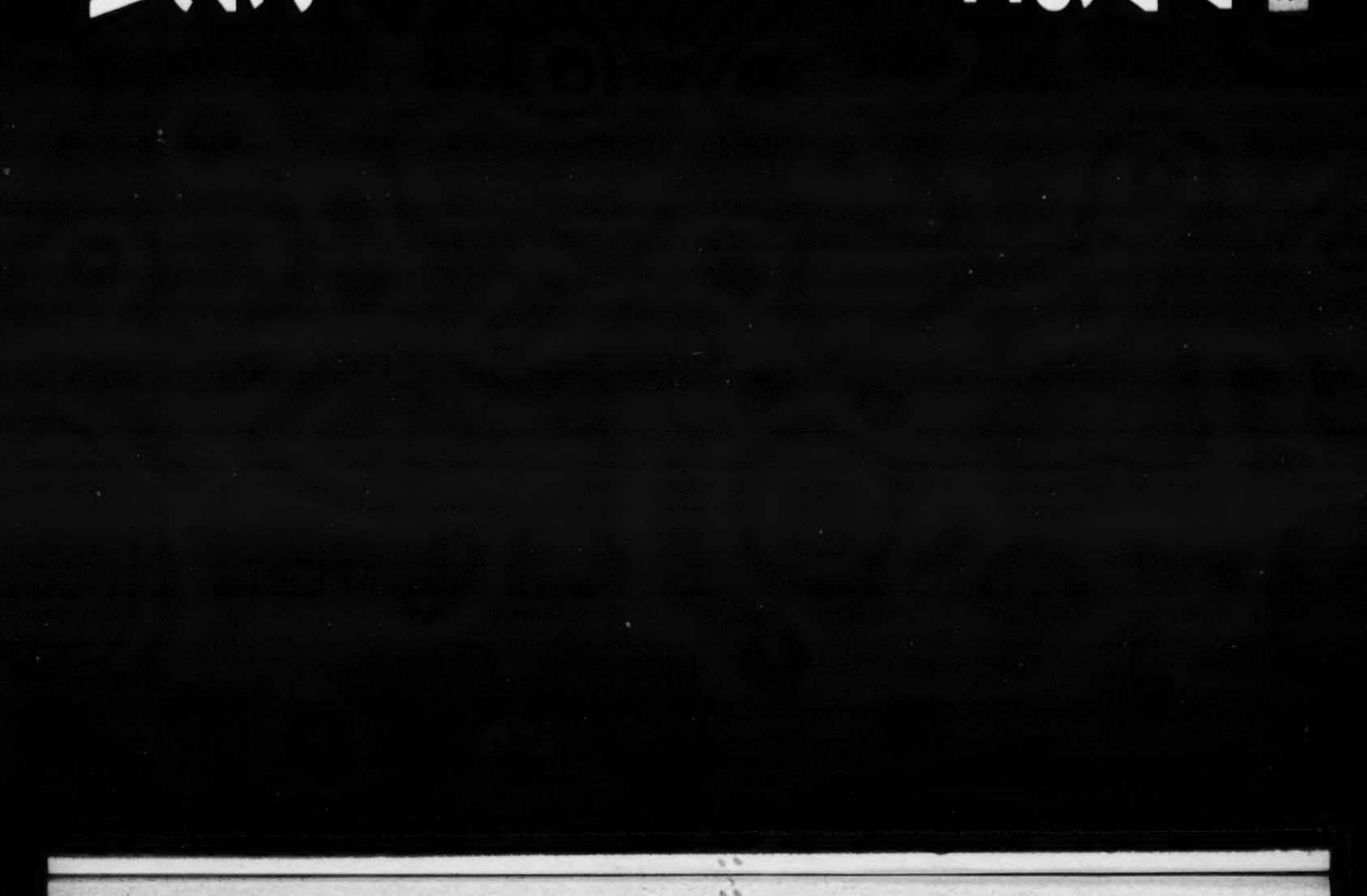
Coefficients in the expansion $\Delta_h^m f(a) \sim \sum_{s=m}^{\infty} B_{m,s}\, h^s D^s f(a)$

m \ s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	s \ m
1	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{120}$	$\frac{1}{720}$	$\frac{1}{5040}$	$\frac{1}{40320}$	$\frac{1}{362880}$	$\frac{1}{3628800}$	$\frac{1}{39916800}$	$\frac{1}{479001600}$	$\frac{1}{6227020800}$	$\frac{1}{87178291200}$	$\frac{1}{1307674368000}$	$\frac{1}{20922789888000}$	$\frac{1}{355687428096000}$	$\frac{1}{6402373705728000}$	$\frac{1}{121645100408832000}$	$\frac{1}{2432902008176640000}$	1
2		1	1	$\frac{7}{12}$	$\frac{1}{4}$	$\frac{31}{360}$	$\frac{1}{40}$	$\frac{127}{20160}$	$\frac{17}{13096}$	$\frac{73}{259200}$	$\frac{31}{604800}$	$\frac{2047}{239500800}$	$\frac{1}{760320}$	$\frac{8191}{43589145600}$	$\frac{5461}{217945728000}$	$\frac{4681}{1494484992000}$	$\frac{257}{697426329600}$	$\frac{131071}{3201186852864000}$	$\frac{73}{16937496576000}$	$\frac{524287}{121645100408320000}$	2
3			1	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{3}{4}$	$\frac{43}{120}$	$\frac{23}{160}$	$\frac{605}{12096}$	$\frac{311}{20160}$	$\frac{2501}{604800}$	$\frac{437}{403200}$	$\frac{115}{456192}$	$\frac{289}{5322240}$	$\frac{2375101}{217945728000}$	$\frac{1190261}{581188608000}$	$\frac{50489}{139485265920}$	$\frac{14039}{232475443200}$	$\frac{132237}{13857951744000}$	$\frac{727577}{508124897280000}$	3
4				1	2	$\frac{13}{6}$	$\frac{5}{3}$	$\frac{81}{80}$	$\frac{37}{72}$	$\frac{9631}{30240}$	$\frac{265}{3024}$	$\frac{55501}{1814400}$	$\frac{253}{25920}$	$\frac{7613}{2661120}$	$\frac{3103}{3991680}$	$\frac{171796901}{871782912000}$	$\frac{4084337}{87178291200}$	$\frac{2532857}{241416806400}$	$\frac{99599}{44834549760}$	$\frac{113363609}{254062448640000}$	4
5					1	$\frac{5}{2}$	$\frac{10}{3}$	$\frac{25}{8}$	$\frac{331}{144}$	$\frac{45}{32}$	$\frac{2343}{3024}$	$\frac{1045}{3024}$	$\frac{7501}{51840}$	$\frac{3600}{45361}$	$\frac{19301}{997920}$	$\frac{8923}{1419264}$	$\frac{332514803}{174356582400}$	$\frac{113561159}{209227898880}$	$\frac{6527611}{44834549760}$	$\frac{220907}{5077939968}$	5
6						1	3	$\frac{19}{4}$	$\frac{21}{4}$	$\frac{1067}{240}$	$\frac{259}{80}$	$\frac{30063}{15120}$	$\frac{97}{90}$	$\frac{63373}{120960}$	$\frac{6671}{28800}$	$\frac{5009023}{53222400}$	$\frac{89809}{2534400}$	$\frac{6511014767}{523069747200}$	$\frac{102179213}{24908083200}$	$\frac{238063743}{186810634000}$	6
7							1	$\frac{7}{2}$	$\frac{77}{12}$	$\frac{49}{6}$	$\frac{1939}{240}$	$\frac{4753}{720}$	$\frac{1249}{270}$	$\frac{77}{27}$	$\frac{136111}{86400}$	$\frac{273217}{345600}$	$\frac{2769673}{7603200}$	$\frac{212737}{1368576}$	$\frac{430187063}{6793113600}$	$\frac{1232148833}{53374464000}$	7
8								1	4	$\frac{25}{2}$	12	$\frac{4819}{360}$	$\frac{123}{10}$	$\frac{1828}{189}$	$\frac{2104}{315}$	$\frac{4992457}{1209600}$	$\frac{77761}{23600}$	$\frac{28513237}{23950080}$	$\frac{41891}{73920}$	$\frac{46068831773}{186810634000}$	8
9									1	$\frac{9}{2}$	$\frac{21}{2}$	$\frac{135}{8}$	$\frac{419}{20}$	$\frac{171}{8}$	$\frac{46943}{2520}$	$\frac{12755}{896}$	$\frac{186653}{19200}$	$\frac{107841}{17920}$	$\frac{18172813}{8322240}$	$\frac{605399}{337920}$	9
10										1	5	$\frac{155}{12}$	$\frac{275}{12}$	$\frac{94}{3}$	$\frac{1265}{36}$	$\frac{135493}{4032}$	$\frac{113465}{4032}$	$\frac{15306213}{725760}$	$\frac{2056713}{145152}$	$\frac{40265267}{4561920}$	10
11											1	$\frac{11}{2}$	$\frac{187}{12}$	$\frac{121}{4}$	$\frac{8129}{180}$	$\frac{8833}{160}$	$\frac{1158509}{20160}$	$\frac{3164029}{60480}$	$\frac{30630009}{725760}$	$\frac{10778317}{345600}$	11
12												1	6	$\frac{37}{2}$	39	$\frac{15149}{240}$	$\frac{3341}{40}$	$\frac{2842373}{30240}$	$\frac{93145}{1008}$	$\frac{13983487}{172800}$	12

Coefficients in the expansion $\Delta_h^m f(a) \sim \sum_{s=m}^{2n} B_{m,s}\, h^s D^s f(a)$—*Concluded*

m \ s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	s \ m
13													1	$\frac{13}{2}$	$\frac{65}{3}$	$\frac{1183}{24}$	$\frac{6877}{80}$	$\frac{176267}{1440}$	$\frac{223873}{1512}$	$\frac{1350479}{8640}$	13
14														1	7	$\frac{301}{12}$	$\frac{245}{4}$	$\frac{82439}{720}$	$\frac{2793}{16}$	$\frac{975623}{4320}$	14
15															1	$\frac{15}{2}$	$\frac{115}{4}$	75	$\frac{7181}{48}$	$\frac{1945}{8}$	15
16																1	8	$\frac{98}{3}$	$\frac{272}{3}$	$\frac{11533}{60}$	16
17																	1	$\frac{17}{2}$	$\frac{221}{6}$	$\frac{867}{8}$	17
18																		1	9	$\frac{165}{4}$	18
19																			1	$\frac{19}{2}$	19
20																				1	20